Exercises for
The Physics of Galaxy Clusters (WS2017/18)

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Exercise sheet 1

Due: Nov. 28, 2017

1. **The Mass Function**

   Masses and length scales are related by the mean background density

$$M = \frac{4\pi}{3}\bar{\rho}R^3. \tag{1}$$

   The mass M_* corresponding to the scale R_* on which the variance becomes unity

$$\sigma_*^2 = 4\pi \int_0^{k_*} \frac{k^2 \mathrm{d}k}{(2\pi)^3} P_\delta(k) = 1 \tag{2}$$

   is called the *nonlinear mass*.

   (a) Show that, for a power-law power spectrum with index n, $P_\delta(k) = Ak^n$, the variance can be written in the form

$$\sigma^2 = \left(\frac{M_*}{M}\right)^{1+n/3}. \tag{3}$$

   (b) Discuss the special cases $n = 1$ and $n = -3$ and draw qualitatively σ^2 as a function of M by interpolating the asymptotic cases of the linear power spectrum.

   (c) Use this result to bring the Press-Schechter mass function into the form

$$f(M,a)\mathrm{d}M \equiv \frac{\partial n_{\mathrm{PS}}(M,a)}{\partial M}\mathrm{d}M = \sqrt{\frac{2}{\pi}}\frac{\alpha\bar{\rho}\delta_{\mathrm{c}}}{M_* D_+} m^{\alpha-2} \exp\left(-\frac{\delta_{\mathrm{c}}^2}{2D_+^2}\, m^{2\alpha}\right)\mathrm{d}m, \tag{4}$$

   where $m = M/M_*$ and $\alpha = 1/2 + n/6$.

2. **The Nonlinear Mass**

   (a) Using *Planck* data, the latest cosmic microwave background measurements yield $\sigma_8 \approx 0.8$ (i.e., σ on the scale of $8\,h^{-1}$ Mpc). Assume that $n = -1$ and estimate the nonlinear mass today.

   (b) How does the nonlinear mass evolve with time?

   (c) Calculate the present abundance of objects with mass M_* according to the Press-Schechter mass function (use $\Omega_{\mathrm{m},0} = 0.3$). Assuming, for simplicity, that these halos are randomly distributed through space, estimate the mean separation between these objects. Compare your estimate with the actual distance of the Milky Way to the Virgo cluster.

3. **Binding Energy**

Consider a dark-matter halo with NFW density profile, i.e.,

$$\rho(r) = \frac{\rho_\mathrm{s}}{x(1+x)^2} \quad \text{with} \quad x = \frac{r}{r_\mathrm{s}}. \tag{5}$$

(a) Using physical reasoning, argue why the potential energy of the halo must be of the form

$$E_\mathrm{pot} = -\alpha \frac{GM_\mathrm{s}^2}{r_\mathrm{s}} \quad \text{with} \quad M_\mathrm{s} = 4\pi r_\mathrm{s}^3 \rho_\mathrm{s}, \tag{6}$$

where $\alpha > 0$ is a dimensionless constant.

(b) Confirm that the gravitational potential of an NFW halo is

$$\Phi(r) = -\frac{GM_\mathrm{s}}{r_\mathrm{s}} \frac{\ln(1+x)}{x}. \tag{7}$$

(c) Determine α by integrating to infinity for simplicity.

4. **Gas in an NFW Halo**

The NFW density profile diverges in the center, i.e., for $x \to 0$. Gas filled in the halo's gravitational potential Φ satisfies Euler's equation

$$\frac{\nabla P_\mathrm{gas}}{\rho_\mathrm{gas}} = -\nabla\Phi(r), \tag{8}$$

where P_gas is the gas pressure.

(a) Assuming an isothermal and ideal gas, show that the gas density profile is

$$\rho_\mathrm{gas} = A \exp\left(-\frac{\bar{m}\Phi}{kT}\right), \tag{9}$$

where T is the temperature, k is Boltzmann's constant, $\bar{m}$ is the mean particle mass, and A is a constant.

(b) Using Equation (7), show that

$$-\frac{\bar{m}\Phi}{kT} = 3\frac{\ln(1+x)}{x} \tag{10}$$

if the gas is in equilibrium with the gravitational potential.

(c) Is the gas density finite in the halo's center? Compare the density profiles of gas and dark matter and explain the differences.

(d) What happens to the gas-to-dark matter mass density ratio ρ_gas/ρ at large radii? Is this a realistic behavior and if not, what would have to be changed in the model to make it more realistic?