Exercises for
The Physics of Galaxy Clusters (WS2017/18)

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Exercise sheet 2

Due: Dec. 19, 2017

1. **Sound Waves**

   By performing a perturbation analysis of the mass and momentum conservation equations, derive the dispersion relation for sound waves,

   $$\omega^2 = \frac{\delta \hat{P}}{\delta \hat{\rho}} k^2, \tag{1}$$

   where the hat indicates Fourier components. Using this dispersion relation, derive the phase and group speed of sound waves. Compare and discuss the different properties of sound and gravity waves.

2. **Gravity Waves**

   Let the total gas pressure and density be related by the isothermal sound speed, $P = c^2_{\mathrm{iso},0}\,\rho$ and let's assume a fixed gravitational field of the form

   $$\boldsymbol{g} = -\frac{g_0}{1 + z/z_0}\boldsymbol{e}_z. \tag{2}$$

   (a) Assuming hydrostatic equilibrium, derive the density stratification, i.e., $\rho = \rho(z, h)$ where $h = c^2_{\mathrm{iso},0}/g_0$ is the pressure scale height.

   (b) Take the limit $z_0 \to \infty$ and compute $\rho(z, h)$.

   (c) Now, compute the Brunt-Väisälä frequency for both atmospheres (finite z_0 and $z_0 \to \infty$). In which of the two atmospheres do you get g-mode trapping and at which height are they trapped?

   (d) Compute the Brunt-Väisälä frequency in the central cluster regions (which have a cuspy NFW density profile) and in the Earth's atmosphere to order of magnitude. To this end, you may take the limit $z \ll z_0$.

3. **Shocks**

Show, that a Galilean transformation of the Rankine-Hugoniot shock jump conditions from the shock to the laboratory rest system leads to the generalized Rankine-Hugoniot conditions of mass, momentum, and energy conservation at a shock,

$$\begin{aligned} \upsilon_\mathrm{s}[\rho] &= [\rho\upsilon], \\ \upsilon_\mathrm{s}[\rho\upsilon] &= [\rho\upsilon^2 + P], \\ \upsilon_\mathrm{s}\left[\rho\frac{\upsilon^2}{2} + \varepsilon\right] &= \left[\left(\rho\frac{\upsilon^2}{2} + \varepsilon + P\right)\upsilon\right]. \end{aligned} \tag{3}$$

Here υ_s and υ denote the shock and the mean gas velocity measured in the laboratory rest system and we introduced the abbreviation $[F] = F_i - F_j$ for the jump of some quantity F across the shock.

4. **Turbulence**

Consider the Navier-Stokes equation in the following compact form

$$\frac{\mathrm{d}\boldsymbol{\upsilon}}{\mathrm{d}t} = -\frac{\boldsymbol{\nabla}P}{\rho} + \nu\boldsymbol{\nabla}^2\boldsymbol{\upsilon}, \tag{4}$$

where $\mathrm{d}/\mathrm{d}t = \partial/\partial t + \boldsymbol{\upsilon}\cdot\boldsymbol{\nabla}$ is the Lagrangian time derivative and $\nu = \eta/\rho$ is the kinematic viscosity.

(a) By introducing characteristic length L_0, velocity V_0, and density ρ_0 scales, rewrite the Navier-Stokes equation into dimensionless form. *Hint:* you also have to introduce a dimensionless time and a dimensionless Nabla operator using these three characteristic scales.

You will find, that the dimensionless equation involves one number, the Reynolds number $\mathrm{Re} \equiv L_0 V_0/\nu$, that characterizes the flow and determines the structure of the solutions to this equation. What is the meaning of this number?

(b) In the lectures, we introduced an energy flow rate per unit mass, $\dot{\epsilon} = \upsilon_\lambda^3/\lambda$, that is valid on all scales λ and constant (because energy does not accumulate at any intermediate scale). Hence, $\dot{\epsilon} = V^3/L$ has also the meaning of an energy injection rate into the turbulent cascade at the outer scale L. Defining the *Kolmogorov length* ℓ, show that this defines corresponding velocity and time scales,

$$\ell \equiv \left(\frac{\nu^3}{\dot{\epsilon}}\right)^{1/4}, \quad \upsilon_\ell = (\dot{\epsilon}\nu)^{1/4}, \quad \tau_\ell = \left(\frac{\nu}{\dot{\epsilon}}\right)^{1/2}. \tag{5}$$

What value has the Reynolds number Re at the *Kolmogorov length* ℓ and why?

Work out the scaling of the following ratios, L_0/ℓ, V_0/υ_ℓ, τ/τ_ℓ, and ϵ_0/ϵ_ℓ with the Reynolds number. We speak about a turbulent flow if the Reynolds number at the outer scale is $\mathrm{Re}(L) \gtrsim 10^3$. Interpret your ratios in the light of this requirement.

(c) Do we have turbulent flows in the hot ($k_\mathrm{B}T = 10$ keV) intracluster medium ($n = 10^{-3}\,\mathrm{cm}^{-3}$), if the outer scale is $L_0 = 300$ kpc and we consider it to be a purely hydrodynamical system? Argue qualitatively, what you would expect to change, if we did add magnetic fields to the system.