Exercises for
The Physics of Galaxy Clusters (WS2017/18)

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Exercise sheet 3

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1. **Cluster Mergers**

   A galaxy group that merges with a galaxy cluster modifies the (thermo-)dynamics of the parent cluster. We approximate the group as a homogeneous sphere of radius R that moves with constant velocity $\boldsymbol{u}$ subsonically in an incompressible fluid (i.e., $\rho = \text{const}$).

   (a) Show that under these assumptions, the velocity potential Φ obeys the Laplace equation, $\Delta\Phi = 0$, where $\boldsymbol{v} = \boldsymbol{\nabla}\Phi$.

   (b) Find the velocity potential Φ and the velocity field $\boldsymbol{v}$ in the cluster rest system.

   (c) Explain qualitatively what changes if the galaxy group moves supersonically?

2. **Adiabatic Cosmic Rays**

   Introducing the dimensionless momentum $p = P_\mathrm{p}/(m_\mathrm{p}\, c)$, we assume that the differential cosmic ray (CR) particle momentum spectrum per volume element can be approximated by a single momentum power law above the minimum momentum q:

   $$f(p) = \frac{\mathrm{d}^2 N}{\mathrm{d}p\,\mathrm{d}V} = C\, p^{-\alpha}\, \theta(p-q), \tag{1}$$

   where $\theta(x)$ denotes the Heaviside step function. The CR pressure is then given by

   $$P_\mathrm{cr} = \frac{m_\mathrm{p} c^2}{3} \int_0^\infty \mathrm{d}p\, f(p) \beta\, p = \frac{C\, m_\mathrm{p} c^2}{6}\, \mathcal{B}_{\frac{1}{1+q^2}}\left(\frac{\alpha-2}{2}, \frac{3-\alpha}{2}\right), \tag{2}$$

   where $\beta = \upsilon/c = p/\sqrt{1+p^2}$ is the dimensionless velocity of the CR particle and $\mathcal{B}_x(a,b)$ denotes the incomplete beta function, and $\alpha > 2$ is assumed.

   (a) Using Liouville's theorem, work out how the low-momentum cutoff q and CR normalization C changes upon an adiabatic density change from ρ_0 to ρ.

   (b) Calculate the CR adiabatic index $\gamma_\mathrm{cr} = \mathrm{d}\ln P_\mathrm{cr}/\mathrm{d}\ln\rho$ and take the non-relativistic limit ($q \ll 1$ and $\alpha > 3$) and the ultra-relativistic limit ($q \to \infty$) of γ_cr.

   (c) Imagine that CRs are accelerated at a strong cosmological structure formation shock with a relative CR pressure of $X_\mathrm{cr} = P_\mathrm{cr}/P_\mathrm{th} = 0.1$. Calculate X_cr in the ultra-relativistic limit after the composite of CRs and thermal gas has experienced adiabatic density increase by a factor of 10^3 from the warm-hot intergalactic medium to the cluster center.

3. **Diffusive Shock Acceleration**

   Restricting to one spatial dimension, the steady-state CR transport equation for the isotropic CR distribution function $f(x, p)$ reads in the limit of negligible Fermi-II acceleration and radiative losses

$$v(x)\frac{\partial f}{\partial x} - \frac{1}{3}\frac{\partial v}{\partial x}p\frac{\partial f}{\partial p} = \frac{\partial}{\partial x}\left[D(x, p)\frac{\partial f}{\partial x}\right], \tag{3}$$

   where D is the CR diffusion coefficient and $v(x)$ is the mean gas velocity. We assume a sharp shock transition of the velocity field (as seen in the shock frame),

$$v(x) = v_1 + (v_2 - v_1)\theta(x), \tag{4}$$

   where the subscripts 1 and 2 denote the gas velocity in the up- and downstream region of the shock. Assume a pre-existing relativistic population in the upstream, f_1, and solve the CR transport equation for $f_2(p)$ in the downstream region of the shock.

4. **Magnetic Force and Stress**

   (a) Show that the Lorentz force can be written in the following ways

$$\boldsymbol{F}_\mathrm{L} = \frac{1}{c}\boldsymbol{j}\times\boldsymbol{B} = \frac{1}{4\pi}(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B} = -\frac{1}{8\pi}\boldsymbol{\nabla}B^2 + \frac{1}{4\pi}(\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{B} = -\boldsymbol{\nabla}\cdot\bar{\boldsymbol{M}}, \tag{5}$$

   where

$$M_{ij} = \frac{1}{8\pi}B^2\delta_{ij} - \frac{1}{4\pi}B_iB_j \tag{6}$$

   is the magnetic stress tensor. It plays a role analogous to the fluid pressure in ordinary fluid mechanics (explaining the minus sign introduced in its definition). The first term on the right of (5) is the gradient of the *magnetic pressure* $B^2/8\pi$ and the second term is often called the magnetic *curvature or tension force* (which can also be present if the field lines are straight).

   (b) Show that the surface force (per unit area) exerted *by* a bounded volume V *on* its surroundings is given by

$$\boldsymbol{F}_\mathrm{S} = \boldsymbol{n}\cdot\bar{\boldsymbol{M}} = \frac{1}{8\pi}B^2\boldsymbol{n} - \frac{1}{4\pi}\boldsymbol{B}B_n, \tag{7}$$

   where $B_n = \boldsymbol{B}\cdot\boldsymbol{n}$ is the component of $\boldsymbol{B}$ along the outward normal $\boldsymbol{n}$ to the surface of the volume.

   (c) To understand the meaning of magnetic stress, take a uniform magnetic field along the z-direction and compute the surface forces $\boldsymbol{F}_\mathrm{S}$ exerted by a rectangular volume that is aligned with the magnetic field (while there are 6 different surface elements, symmetry limits the surface forces to only two different types). Which magnetic force terms (pressure or tension) are contributing to these surface forces? Explain the action of these forces and why magnetic fields can be thought of as elastic wires.