Reading Assignment for
The Physics of Galaxy Clusters (Summer Term 2020)

Lecturer: Christoph Pfrommer

in preparation of lecture 6

Next online meeting May 26, 2020, 12:00

Please read and work through the script, covering the following topics:

3.1.3 Buoyancy Instabilities

3.1.4 Vorticity

3.1.5 Turbulence

I prepared the following questions that should help you to understand the topics. Please read a topic first, think about it and then work through my set of questions on this topic. Some questions are going beyond what you have read in the lecture notes (indicated by *Bonus* questions). I do not expect you to answer these questions as well, but I would like you to start thinking about them and they will certainly be the starting point for our next zoom meeting. Ideally you can come up with many more questions yourself!

- **Buoyancy Instabilities – Perturbation analysis on a hydrostatic background**
  - Derive the most general background temperature profile of a hydrostatic background. Is this a strict constraint or can you violate it? If so, under which condition?
  - Derive the first-order conservation equations (3.60) – (3.62).
  - Transform this into Fourier space by decomposing all dynamical variables into plane waves and verify Eqns. (3.64) – (3.66).
  - Why is the gas nearly incompressible? What are the underlying approximations?
  - Project the momentum equation (3.65) into a purely vertical and perpendicular part.
  - Use these projected equations and the perturbed entropy equation (3.66) to derive the dispersion relation for gravity waves.
  - Discuss the solution of this dispersion relation. Why do g-modes exhibit a maximum possible frequency?
  - *Bonus:* Derive the entropy condition for a stably stratified atmosphere (the "Schwarzschild condition") purely through thermodynamical considerations. How do these two derivations differ in their assumptions? Which result teaches us more about the stratified atmosphere and why?
- **Vorticity**
  - Why can an incompressible vector field be described as a pure vortex field?
  - Derive Eqns. (3.74) and (3.75).
  - Derive the evolution equation for vorticity in general as well as in the small-velocity limit, $\mathcal{M} \ll 1$.
  - Explain why vorticity (in the approximation $\mathcal{M} \ll 1$) is principally generated in the horizontal plane (perpendicular to gravity).

- **Turbulence**
  - Derive Eqn. (3.78) from the general Navier-Stokes equation (3.45).
  - Explain the physical meaning of the Reynolds number.
  - Explain the energy cascade and derive the steady-state scaling of velocity v_λ with scale λ.
  - Why does this imply strong intermittency on small scales? *Bonus:* Can you describe a turbulent velocity field by a purely Gaussian process?
  - Derive the Kolmogorov turbulent power spectrum of driven turbulence.

- **Sound Waves**
  1. By performing a perturbation analysis of the mass and momentum conservation equations, derive the dispersion relation for sound waves,
  $$\omega^2 = \frac{\delta \hat{P}}{\delta \hat{\rho}} k^2, \tag{1}$$
  where the hat indicates Fourier components.
  2. Using this dispersion relation, derive the phase and group speed of sound waves.
  3. Compare and discuss the different properties of sound and gravity waves.

- **Gravity Waves**

  Let the total gas pressure and density be related by the isothermal sound speed, $P = c_{\mathrm{iso},0}^2\, \rho$ and let's assume a fixed gravitational field of the form
  $$\vec{g} = -\frac{g_0}{1 + z/z_0} \vec{e}_z. \tag{2}$$
  1. Assuming hydrostatic equilibrium, derive the density stratification, i.e., $\rho = \rho(z, h)$ where $h = c_{\mathrm{iso},0}^2 / g_0$ is the pressure scale height.
  2. Take the limit $z_0 \to \infty$ and compute $\rho(z, h)$.
  3. Now, compute the Brunt-Väisälä frequency for both atmospheres (finite z_0 and $z_0 \to \infty$). In which of the two atmospheres do you get g-mode trapping and at which height are they trapped?
  4. Compute the Brunt-Väisälä frequency in the central cluster regions (which have a cuspy NFW density profile) and in the Earth's atmosphere to order of magnitude. To this end, you may take the limit $z \ll z_0$.