Reading Assignment for
The Physics of Galaxy Clusters (Summer Term 2020)

Lecturer: Christoph Pfrommer

in preparation of lecture 7

Next online meeting June 2, 2020, 12:00

Please read and work through the script, covering the following topic:

3.1.6 Shocks

I prepared the following questions that should help you to understand the topics. Please read a topic first, think about it and then work through my set of questions on this topic. Some questions are going beyond what you have read in the lecture notes (indicated by *Bonus* questions). I do not expect you to answer these questions as well, but I would like you to start thinking about them and they will certainly be the starting point for our next zoom meeting. Ideally you can come up with many more questions yourself!

- **Shocks**
  - Explain how you can generate a shock.
  - If a shock is a true discontinuity in hydrodynamic quantities, are our partial derivatives in our evolution equations for mass, momentum and energy well defined at the shock? How does one deal with this issue in practice? What is really happening at a shock?
  - Derive an evolution equation for the kinetic energy density $\rho v^2/2$.
  - Start at the conservation equations (3.90) to (3.92) and derive the Rankine-Hugoniot jump conditions in the shock rest frame, i.e. derive Eqns. (3.108) to (3.110).
  - How does a tangential discontinuity differ from a shock? Name at least two of the three differences.
  - What is the physical interpretation of the Mach number?
  - Derive the Rankine-Hugoniot jump conditions for strong shocks of Eqns. (3.118) to (3.120). Why does the density only jump by a factor of 4?
  - Why is energy conservation apparently violated across the shock in the shock rest frame (see Eqn. 3.123)? How is this conundrum relieved?
  - Verify the statement "A shock converts supersonic gas into denser, slower moving, higher pressure, subsonic gas" through equations (for simplicity, you can use the strong-shock limit $\mathcal{M}_1 \gg 1$).
  - Explain the difference between an adiabatic curve and a shock adiabat in the $P - V$ diagram.
  - Explain what changes qualitatively at an oblique shock in comparison to a plane-parallel shock.
- **Generalized Rankine-Hugoniot Shock Jumps**

  Show, that a Galilean transformation of the Rankine-Hugoniot shock jump conditions from the shock to the laboratory rest system leads to the generalized Rankine-Hugoniot conditions of mass, momentum, and energy conservation at a shock,

$$
\begin{aligned}
v_{\mathrm{s}}[\rho] &= [\rho v], \\
v_{\mathrm{s}}[\rho v] &= [\rho v^2 + P], \\
v_{\mathrm{s}}\left[\rho\frac{v^2}{2} + \varepsilon\right] &= \left[\left(\rho\frac{v^2}{2} + \varepsilon + P\right) v\right].
\end{aligned}
\tag{1}
$$

Here v_{s} and v denote the shock and the mean gas velocity measured in the laboratory rest system, $\varepsilon = \epsilon\rho$ is the thermal energy density, and we introduced the abbreviation $[F] = F_i - F_j$ for the jump of some quantity F across the shock.

- **Turbulent Scaling Laws**

Consider the Navier-Stokes equation in the following compact form

$$
\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\frac{\boldsymbol{\nabla} P}{\rho} + \nu \boldsymbol{\nabla}^2 \boldsymbol{v}, \tag{2}
$$

where $\mathrm{d}/\mathrm{d}t = \partial/\partial t + \boldsymbol{v} \cdot \boldsymbol{\nabla}$ is the Lagrangian time derivative and $\nu = \eta/\rho$ is the kinematic viscosity.

1. By introducing characteristic length L_0, velocity V_0, and density ρ_0 scales, rewrite the Navier-Stokes equation into dimensionless form. *Hint:* you also have to introduce a dimensionless time and a dimensionless Nabla operator using these three characteristic scales.

   You will find, that the dimensionless equation involves one number, the Reynolds number $\mathrm{Re} \equiv L_0 V_0/\nu$, that characterizes the flow and determines the structure of the solutions to this equation. What is the meaning of this number?

2. In the lectures, we introduced an energy flow rate per unit mass, $\dot{\epsilon} = v_\lambda^3/\lambda$, that is valid on all scales λ and constant (because energy does not accumulate at any intermediate scale). Hence, $\dot{\epsilon} = V^3/L$ has also the meaning of an energy injection rate into the turbulent cascade at the outer scale L. Defining the *Kolmogorov length* ℓ, show that this defines corresponding velocity and time scales,

$$
\ell \equiv \left(\frac{\nu^3}{\dot{\epsilon}}\right)^{1/4}, \quad v_\ell = (\dot{\epsilon}\nu)^{1/4}, \quad \tau_\ell = \left(\frac{\nu}{\dot{\epsilon}}\right)^{1/2}. \tag{3}
$$

   What value has the Reynolds number Re at the *Kolmogorov length* ℓ and why?

   Work out the scaling of the following ratios, L_0/ℓ, V_0/v_ℓ, τ/τ_ℓ, and ϵ_0/ϵ_ℓ with the Reynolds number. We speak about a turbulent flow if the Reynolds number at the outer scale is $\mathrm{Re}(L) \gtrsim 10^3$. Interpret your ratios in the light of this requirement.

3. Do we have turbulent flows in the hot ($k_{\mathrm{B}}T = 10$ keV) intracluster medium ($n = 10^{-3}\,\mathrm{cm}^{-3}$), if the outer scale is $L_0 = 300$ kpc and we consider it to be a purely hydrodynamical system? Argue qualitatively, what you would expect to change, if we did add magnetic fields to the system.