Reading Assignment for
The Physics of Galaxy Clusters (Winter Term 2022/23)

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in preparation of lecture 7

Next lecture Nov 30, 2022, 14:15

Please read and work through the script, covering the following topic:

3.1.6 Shocks

I prepared the following questions that should help you to understand the topics. Please read a topic first, think about it and then work through my set of questions on this topic. Some questions are going beyond what you have read in the lecture notes (indicated by *Bonus* questions). I do not expect you to answer these questions as well, but I would like you to start thinking about them and they will certainly be the starting point for our next lecture. Ideally you can come up with many more questions yourself!

- **Gravity Waves.** This problem complements the topics *buoyancy instabilities* and *sound waves* from last week.

  Let the total gas pressure and density be related by the isothermal sound speed, $P = c_{\mathrm{iso},0}^2\,\rho$ and let's assume a fixed gravitational field of the form

  $$\boldsymbol{g} = -\frac{g_0}{1+z/z_0}\boldsymbol{e}_z. \qquad (1)$$

  1. Assuming hydrostatic equilibrium, derive the density stratification, i.e., $\rho = \rho(z,h)$ where $h = c_{\mathrm{iso},0}^2/g_0$ is the pressure scale height.
  2. Take the limit $z_0 \to \infty$ and compute $\rho(z,h)$.
  3. Now, compute the Brunt-Väisälä frequency for both atmospheres (finite z_0 and $z_0 \to \infty$). In which of the two atmospheres do you get g-mode trapping and at which height are they trapped?
  4. Compute the Brunt-Väisälä frequency in the central cluster regions (which have a cuspy NFW density profile) and in the Earth's atmosphere to order of magnitude. To this end, you may take the limit $z \ll z_0$.

- **Shocks.** These are the questions for your reading assigment:
  - Explain how you can generate a shock.
  - If a shock is a true discontinuity in hydrodynamic quantities, are our partial derivatives in our evolution equations for mass, momentum and energy well defined at the shock? How does one deal with this issue in practice? What is really happening at a shock?
  - Derive an evolution equation for the kinetic energy density $\rho v^2/2$.
  - Start at the conservation equations (3.93) to (3.95) and derive the Rankine-Hugoniot jump conditions in the shock rest frame, i.e. derive Eqns. (3.111) to (3.113).
  - How does a tangential discontinuity differ from a shock? Name at least two of the three differences.
  - What is the physical interpretation of the Mach number?
  - Derive the Rankine-Hugoniot jump conditions for strong shocks of Eqns. (3.121) to (3.123). Why does the density only jump by a factor of 4?

– Why is energy conservation apparently violated across the shock in the shock rest frame (see Eqn. 3.126)? How is this conundrum relieved?

– Verify the statement "A shock converts supersonic gas into denser, slower moving, higher pressure, subsonic gas" through equations (for simplicity, you can use the strong-shock limit $\mathcal{M}_1 \gg 1$).

– Explain the difference between an adiabatic curve and a shock adiabat in the $P-V$ diagram.

– Explain what changes qualitatively at an oblique shock in comparison to a plane-parallel shock.

- **Generalized Rankine-Hugoniot Shock Jumps**

  Show, that a Galilean transformation of the Rankine-Hugoniot shock jump conditions from the shock to the laboratory rest system leads to the generalized Rankine-Hugoniot conditions of mass, momentum, and energy conservation at a shock,

$$\begin{aligned} v_{\rm s}[\rho] &= [\rho u], \\ v_{\rm s}[\rho u] &= [\rho u^2 + P], \\ v_{\rm s}\left[\rho \frac{u^2}{2} + \varepsilon\right] &= \left[\left(\rho \frac{u^2}{2} + \varepsilon + P\right) u\right]. \end{aligned} \tag{2}$$

  Here $v_{\rm s}$ and u denote the shock and the mean gas velocity measured in the laboratory rest system, $\varepsilon = \epsilon\rho$ is the thermal energy density, and we introduced the abbreviation $[F] = F_i - F_j$ for the jump of some quantity F across the shock.