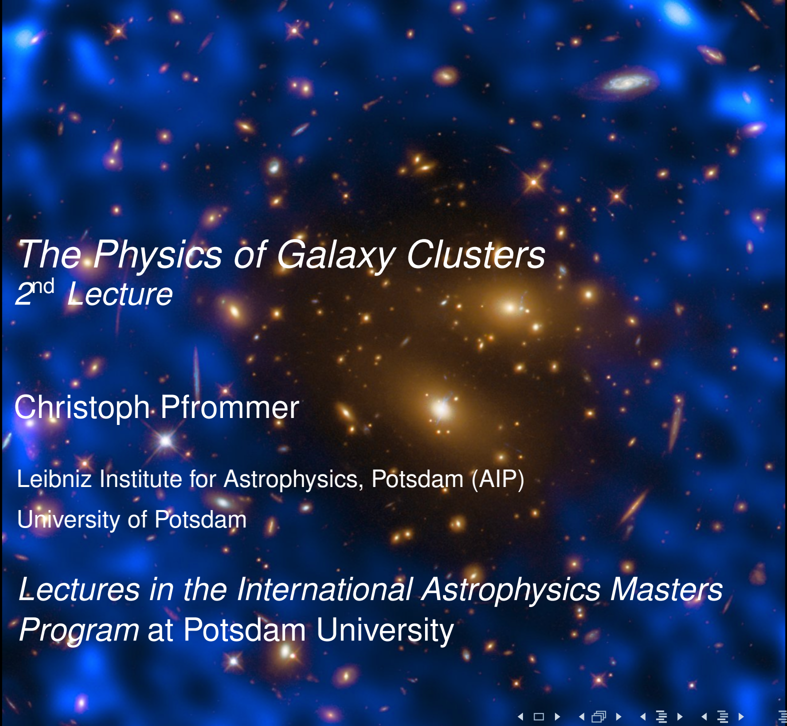

The Physics of Galaxy Clusters

2nd Lecture

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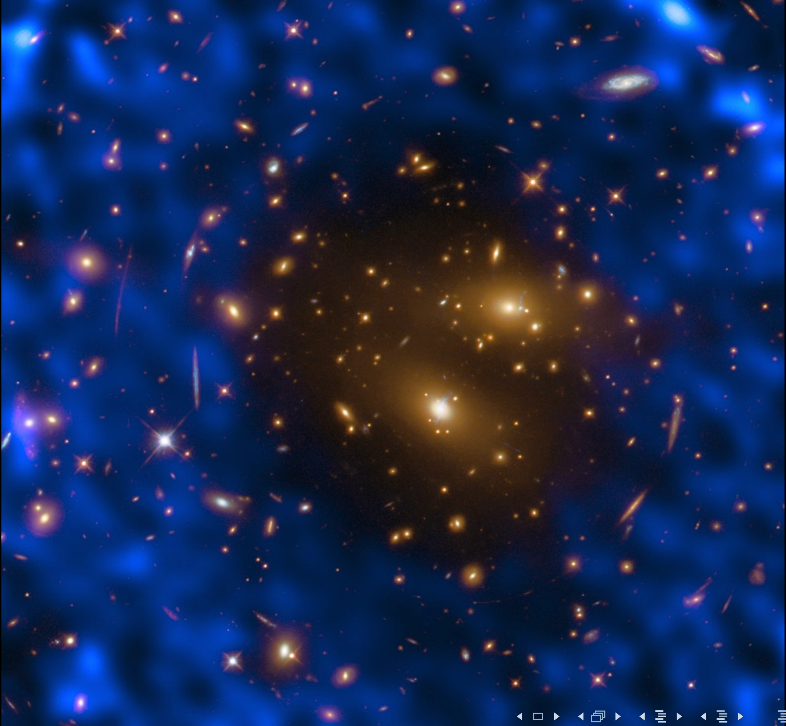

Overview of galaxy clusters: observations and simulations of different observational probes of clusters

- Order of magnitudes
- Optical window: galaxies probe cluster masses; clusters provide a great "magnifying glass" for studying transformational processes of galaxies
- X-ray emission: intracluster plasma probes cluster masses and hydrodynamical flows and instabilities
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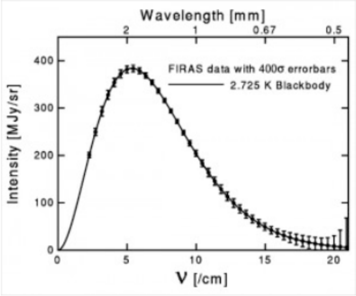
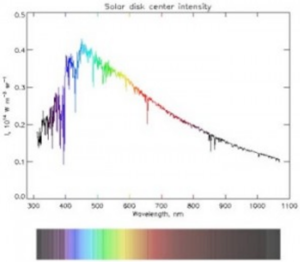
- Sunyaev-Zel'dovich effect
- Relation to the average universe

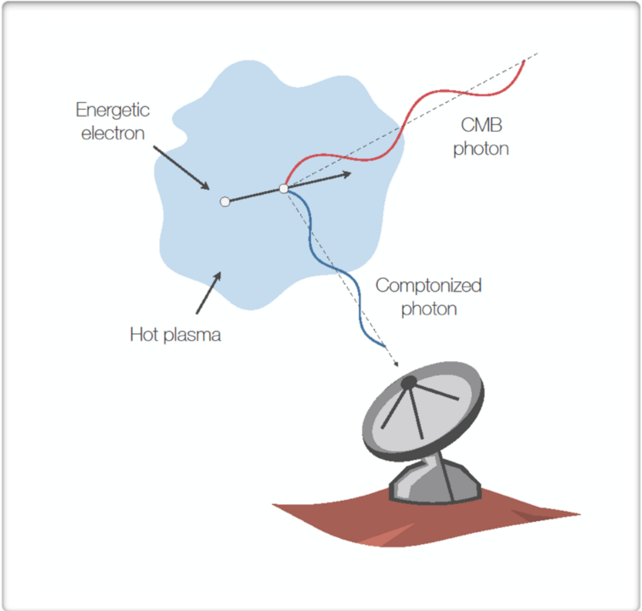
Cosmic microwave background (CMB) spectrum

Comparing the perfect black-body spectrum of the CMB and that of the Sun

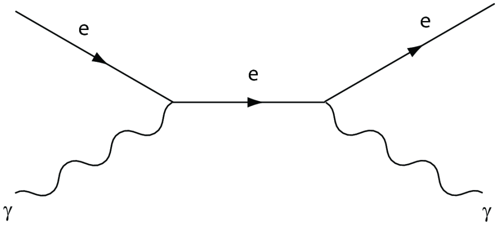




Sunyaev-Zel'dovich effect: idea



Sunyaev-Zel'dovich effect: Compton scattering



- CMB photon interacts with electrons of the ICM via Compton scattering
- elastic scattering event conserves number of CMB photons
- mean energy transfer from the "hot" electron to the "cold" photon: "inverse Compton scattering"
- causes distortion of the CMB spectrum: decrement in thermodynamic temperature below $\nu_0 \approx 220$ GHz, and an excess above
- galaxy clusters appear as holes in the CMB sky at $\nu < \nu_0$ and as extended sources above

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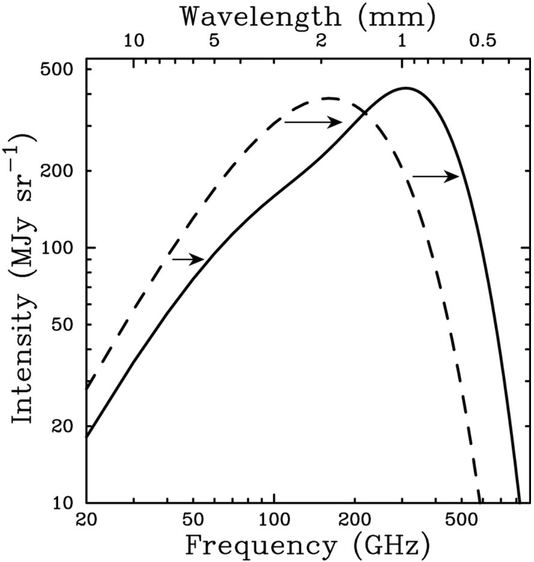
Sunyaev-Zel'dovich effect: spectral shift



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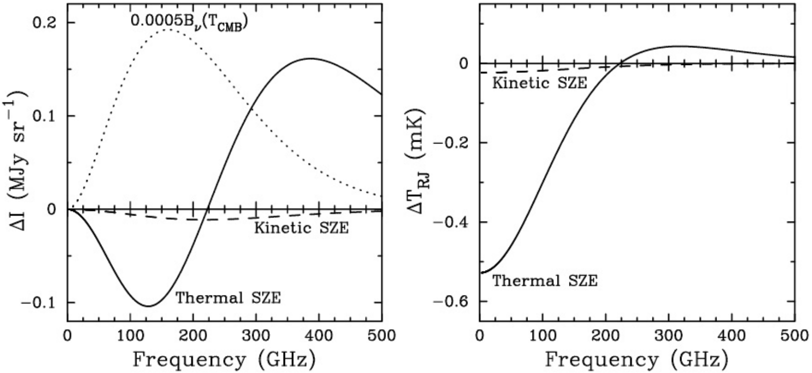

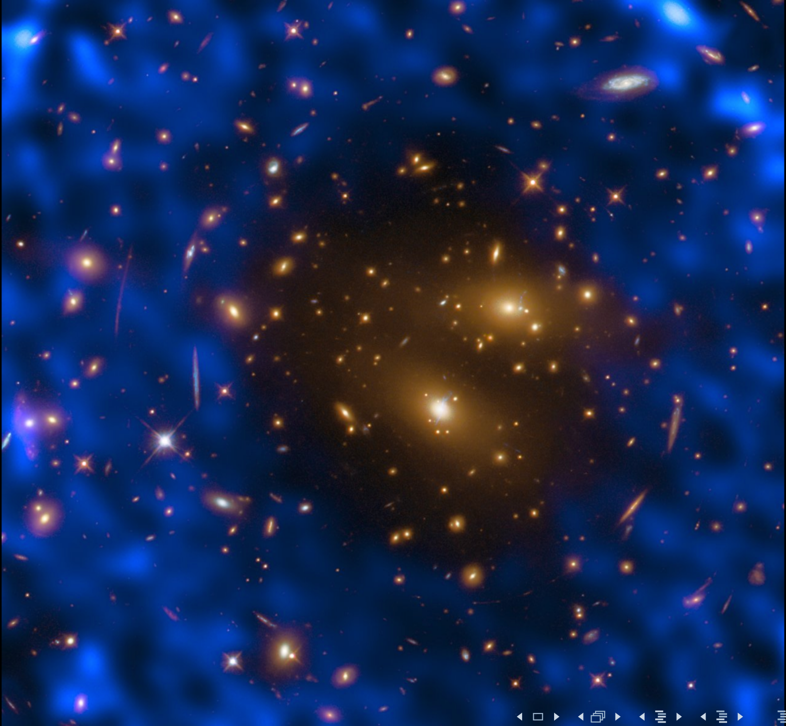

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- To answer this, we compute the optical depth

$$\tau = \int_0^L n_\mathrm{e}\sigma_\mathrm{T}\mathrm{d}l \approx n_\mathrm{e}\sigma_\mathrm{T}L,$$

where L is the effective path length through the hot intracluster medium and σ_T is the Thomson cross section,

$$\sigma_\mathrm{T} \approx 2\pi r_0^2 = 2\pi\left(\frac{e^2}{m_\mathrm{e}c^2}\right)^2 \approx 6\left[\frac{\left(4.8\times10^{-10}\right)^2}{10^{-27}\,10^{21}}\right]^2 \mathrm{cm}^2$$

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- Hence, we obtain an optical depth

$$\tau = n_\mathrm{e}\sigma_\mathrm{T}L \approx 10^{-4}\,\mathrm{cm}^{-3}\,6\times10^{-25}\,\mathrm{cm}^2\,10^{25}\,\mathrm{cm} \approx 6\times10^{-4} \ll 1.$$

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$\Rightarrow$ on average only one photon in 2000 experiences a scattering event

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- This is the definition of the Compton-y parameter,

$$y_\mathrm{cl} = \int_0^L \frac{k_\mathrm{B} T_\mathrm{e}}{m_\mathrm{e} c^2} n_\mathrm{e}\sigma_\mathrm{T}\mathrm{d}l \approx 10^{-2} \times 6 \times 10^{-4} = 6 \times 10^{-6},$$

where, we adopted a line-of-sight averaged temperature of our massive ($10^{15}\,\mathrm{M}_\odot$) cluster with $k_\mathrm{B} T_\mathrm{e} \approx 6$ keV. This is a tiny signal!

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- As we can see, the SZ signal is proportional to the integrated electron pressure ($P_e = k_B T_e n_e$), so the hot gas of the galaxy clusters dominates the effect, but by how much?
- What about the other parts of the photon path? There is the photon propagation through the intergalactic medium (IGM) and the halo of our galaxy ⇒ exercise

Compton-y parameter: $y = \int_0^L \frac{k_B T_e}{m_e c^2} n_e \sigma_T dl \sim y_{cl} + y_{IGM} + y_{MW}$

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Y = \int y \mathrm{d}\Omega &= \frac{1}{D_\mathrm{ang}^2(0,z)} \int y \mathrm{d}^2 R \\
&= \frac{1}{D_\mathrm{ang}^2(0,z)} \frac{\sigma_\mathrm{T}}{m_\mathrm{e} c^2} \int_0^{R_\mathrm{vir}} P_\mathrm{e} \mathrm{d}V \\
&= \frac{\gamma - 1}{D_\mathrm{and}^2(0,z)} \frac{\sigma_\mathrm{T}}{m_\mathrm{e} c^2} x_\mathrm{e} X_\mathrm{H} \mu E_\mathrm{gas}
\end{aligned}
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where $x_\mathrm{e} = n_\mathrm{e}/n_\mathrm{H}$ is the free electron fraction, $X_\mathrm{H} = 0.76$ is the primordial hydrogen mass fraction, and μ is the molecular weight so that

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- Hence, the integrated Compton-y parameter depends on redshift via $Y \propto D_{\mathrm{ang}}^{-2}(0,z)$

Synthesis of observational windows

- Which observational method would you prefer to observe the inner parts of a cluster? which for the outer parts?

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- Which observational method would you prefer to observe the inner parts of a cluster? which for the outer parts?
- Which problem do you see arising for studying small clusters with galaxy observations? How reliable can you estimate the cluster mass here?
- Which method is most powerful to do cluster cosmology? which criteria would you find most important for this?

Relation to average Universe

- The critical density of the universe is the density that closes the universe, i.e., it would make the cosmic expansion asymptotically come to a halt in the absence of a cosmological constant or dark energy. Its value today is given by

$$\rho_{\mathrm{cr},0} = \frac{3H_0^2}{8\pi G} \approx 10^{-29}\ \mathrm{g\ cm^{-3}},$$

  where $H_0 \approx 70$ km s^{-1} Mpc^{-1} is the Hubble constant.
- To order of magnitude, we can show why this form is obvious (without invoking general relativity). If gravity dominates, the dynamical time of a system is $t_{\mathrm{dyn}} \sim 1/\sqrt{G\rho}$. Where does this come from?

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$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -\frac{Gm}{r^2} \sim \frac{r}{t^2} \quad \Rightarrow \quad t_{\mathrm{ff}} \sim \frac{1}{\sqrt{G\rho}}$$

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$$\sim 4 \times 10^{10}\,\mathrm{M}_\odot\ \mathrm{Mpc}^{-3} \sim 10^{9}\,\mathrm{M}_\odot\ \mathrm{Mlyr}^{-3}.$$

  $\Rightarrow$ let's compare this to typical cluster masses $M_{\mathrm{cl}} \sim 10^{15}\,\mathrm{M}_\odot$; in order to form clusters, we need large chunks of volume that contain $10^{15}\,\mathrm{M}_\odot$

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  Hence, we have to increase the radius to $r_{\mathrm{cl}} = 20$ Mpc to collect enough mass to build a cluster. Upon radial collapse by a factor of ~ 10, we obtain a cluster mass of $M \sim 10^{15}\,\mathrm{M}_\odot$.

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- Thus, we typically find cluster densities of $\bar{\rho}_{\mathrm{cl}} \sim 10^3\,\bar{\rho}_{\mathrm{m},0}$ that form as a result of gravitational collapse. This collapse is stopped by the virialization process (next week).

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The Growth of perturbations – methodology

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- We begin with the continuity equation, which formulates mass conservation,

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  where $\rho(t, \boldsymbol{r})$ and $\boldsymbol{v}(t, \boldsymbol{r})$ are the density and velocity of the cosmic fluid at position $\boldsymbol{r}$ and time t.

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- The second equation is Euler's equation which formulates the conservation of momentum,

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla}_{\boldsymbol{r}})\boldsymbol{v} = -\frac{\boldsymbol{\nabla}_{\boldsymbol{r}} P}{\rho} - \boldsymbol{\nabla}_{\boldsymbol{r}} \Phi \,.$$

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- The next steps consist in decomposing density and velocity fields into their homogeneous background values $\bar{\rho}$ and $\bar{\boldsymbol{v}}$ and small perturbations $\delta\rho$ and $\delta\boldsymbol{v}$,

$$\rho(t, \boldsymbol{r}) = \bar{\rho}(t) + \delta\rho(t, \boldsymbol{r}) \ , \quad \boldsymbol{v}(t, \boldsymbol{r}) = \bar{\boldsymbol{v}}(t) + \delta\boldsymbol{v}(t, \boldsymbol{r}) \ .$$

The Growth of perturbations: background evolution

- The evolution of the homogeneous background quantities are governed by the expansion of the universe. Physical coordinates, $\boldsymbol{r}$, are related to comoving coordinates, $\boldsymbol{x}$, via the equation $\boldsymbol{r} = a\boldsymbol{x}$. Here, $a(t)$ is the cosmic scale factor whose dynamics is governed by Friedmann's equations:

$$H^2(a) \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2} + \frac{\Lambda c^2}{3}\,, \qquad (1)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3}\,. \qquad (2)$$

Here, K is a constant parameterizing the curvature of spatial hypersurfaces and Λ is the cosmological constant. The scale factor is uniquely determined once its value at a fixed time t is chosen. We set $a = 1$ today.

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- If we set the radius $r(t) \equiv a(t)R$ where $R = \text{const.}$, we can interpret Friedmann's Equations in terms of Newtonian dynamics, i.e., an energy equation and an equation of motions of a test mass on the surface of a sphere that is embedded in an (infinite) expanding, homogeneous mass density distribution $\rho(t)$ of a idealized pressureless fluid (proof in cosmology lectures):

$$\left(\frac{\dot{r}}{r}\right)^2 = \frac{8\pi G}{3}\rho + \frac{C}{r^2},$$

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The Growth of perturbations: background evolution

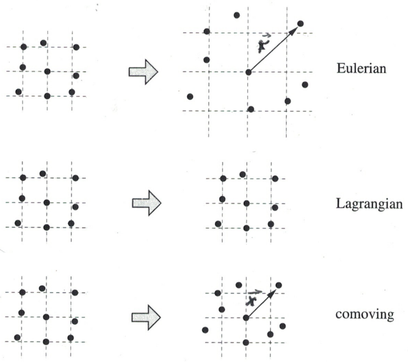
- If we set the radius $r(t) \equiv a(t)R$ where $R = \text{const.}$, we can interpret Friedmann's Equations in terms of Newtonian dynamics, i.e., an energy equation and an equation of motions of a test mass on the surface of a sphere that is embedded in an (infinite) expanding, homogeneous mass density distribution $\rho(t)$ of a idealized pressureless fluid (proof in cosmology lectures):

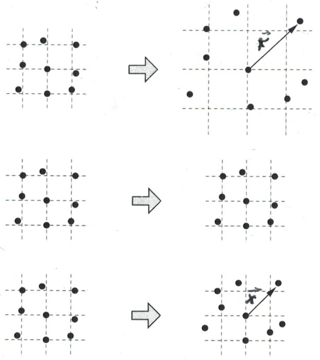
$$\left(\frac{\dot{r}}{r}\right)^2 = \frac{8\pi G}{3}\rho + \frac{C}{r^2}$$

$$\ddot{r} = -\frac{4\pi G}{3} r \left(\rho + \frac{3p}{c^2}\right)$$

- The density sources gravity. The pressure term adds to the density because pressure is a consequence of particle motion, i.e. the kinetic energy of particles, which is equivalent to a mass density and thus acts gravitationally.
- Choosing $K = -C/c^2$ yields the curvature term, i.e., K is the constant of integration and determines the fate of the expansion.
- The Λ term has no analogy in Newtonian dynamics.

The Growth of perturbations: comoving coordinates



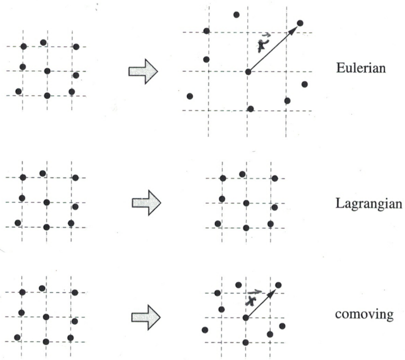


- We transform physical coordinates, $\boldsymbol{r}$, to comoving coordinates, $\boldsymbol{x}$, which are related by $\boldsymbol{r} = a\boldsymbol{x}$.
- We obtain an expression for the velocity,

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = \dot{a}\boldsymbol{x} + a\dot{\boldsymbol{x}} = H\boldsymbol{r} + a\dot{\boldsymbol{x}} = \bar{\boldsymbol{v}} + \delta\boldsymbol{v},$$

where $\bar{\boldsymbol{v}} = H\boldsymbol{r}$ is the Hubble velocity and $\delta\boldsymbol{v} = a\dot{\boldsymbol{x}}$ is the peculiar velocity that deviates from the Hubble flow.

The Growth of perturbations: comoving coordinates



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  where $\bar{\boldsymbol{v}} = H\boldsymbol{r}$ is the Hubble velocity and $\delta\boldsymbol{v} = a\dot{\boldsymbol{x}}$ is the peculiar velocity that deviates from the Hubble flow.
- What is the physical meaning of this separation? Why does this insight help in deriving the perturbation equations?

AIP

The Growth of perturbations – methodology

- We define the density contrast,

$$\delta \equiv \frac{\delta\rho}{\bar{\rho}} \ ,$$

and adopt an equation of state linking the pressure fluctuation to the density fluctuation,

$$\delta p = \delta p(\delta) \equiv c_\mathrm{s}^2 \delta\rho$$

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with the sound speed c_s.
- The equations of mass, momentum, and energy conservation can be combined to yield a single equation for the density contrast
$$\ddot{\delta} + 2H\dot{\delta} = \left(4\pi G\bar{\rho}\delta + \frac{c_s^2 \nabla_{\boldsymbol{x}}^2 \delta}{a^2}\right)\ .$$

AIP

The Growth of perturbations: solutions

- You derived an ordinary differential equation for the density contrast δ:

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- Which mathematical form has this equation? What is the physical meaning of the term with H? How do you solve it?

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- I derived in my notes that the growing and decaying mode solutions in the matter dominated phase read as

$$\hat{\delta} \propto \begin{cases} a\,, \\ a^{-3/2}\,. \end{cases}$$

Which of these are responsible for growing a cluster? which mode grows cosmic voids (huge volumes of nearly empty space)? Are decaying modes responsible for growing these? Justify your answer.

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- In the derivation, the pressure term provides the restoring force. It communicates the pressure gradients via collisions and sound waves to the gas. However, dark matter does not interact via sound waves? How would you need to conceptually change the derivation to account for a collisionless dark matter component? How does dark matter influence structure formation?

AIP

Power spectra

- The variance of δ *in Fourier space* defines the power spectrum $P(k)$,

$$\langle\hat{\delta}(\boldsymbol{k})\hat{\delta}^*(\boldsymbol{k}')\rangle \equiv (2\pi)^3 P(k)\delta_{\mathrm{D}}(\boldsymbol{k}-\boldsymbol{k}') \;,$$

  where δ_{D} is Dirac's delta distribution.
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- derive relation between power spectrum and correlation function and introduce filtering concept on blackboard

- Sunyaev-Zel'dovich Effect
- Synthesis of Observational Windows
- Relation to Average Universe
- The Growth of Perturbations: Newtonian Equations and Density Perturbations
- Power Spectra