The Physics of Galaxy Clusters
3rd Lecture

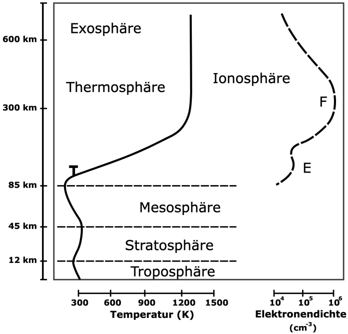
Christoph Pfrommer

Leibniz Institute for Astrophysics, Potsdam (AIP)

University of Potsdam

Lectures in the International Astrophysics Masters Program at Potsdam University

Sunyaev-Zel'dovich effect – ionosphere contribution



Compton-y parameter: $y = \int_0^L \frac{k_B T_e}{m_e c^2} n_e \sigma_T \mathrm{d}l = y_{cl} + y_{IGM} + y_{MW} + y_{ion-sph}$

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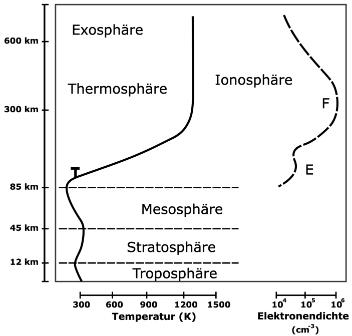

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- cluster: $n_\mathrm{e} \sim 10^{-4}\,\mathrm{cm}^{-3}$, $kT \sim 6$ keV, $L \sim 3$ Mpc:
- ionosphere: $n_\mathrm{e} \sim 10^{6}\,\mathrm{cm}^{-3}$, $kT \sim 0.12$ eV, $L \sim 300$ km:

$$\frac{y_\mathrm{cl}}{y_\mathrm{ion-sph}} = \frac{n_\mathrm{e,cl}}{n_\mathrm{e,ion-sph}} \frac{kT_\mathrm{cl}}{kT_\mathrm{ion-sph}} \frac{L_\mathrm{cl}}{L_\mathrm{ion-sph}}$$
$$\sim 10^{-10} \times 5 \times 10^{4} \times 3 \times 10^{17} \sim 1.5 \times 10^{12}$$

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- The Growth of Perturbations:
  * cluster form through gravitational instability of overdensities:
  $\lambda < \lambda_J$: perturbations oscillate
  $\lambda > \lambda_J$: perturbations grow as $\delta \propto a$ (during matter domination, $\Omega = 1$)
  * constructive interference of large-scale primordial waves grow into clusters
  * destructive interference of primordial waves grow into voids

The Physics of Galaxy Clusters

Recap of last week's lecture

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  * equivalent information content as correlation function (power spectrum is the Fourier transform of the correlation function)

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$\Rightarrow$ today's topic: how do clusters form?

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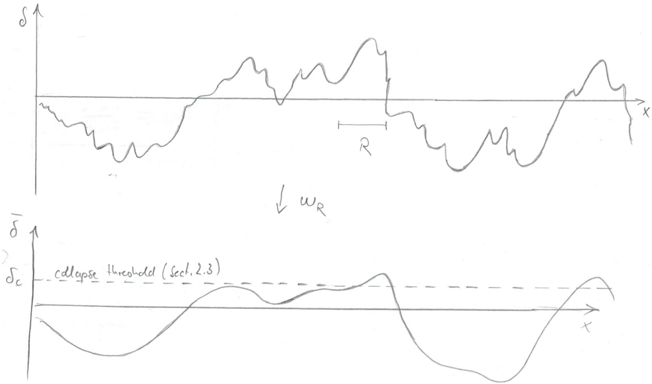
Window function applied to density field

- How does the window function influence the density contrast? Draw some one-dimensional function that varies widely.



Window function: impact on power spectrum

- How does this look after you applied a top-hat or a Gaussian window? Imagine you have field δ with two spatial scales (one with a large and one with a small wavelength). Then you apply the filter which has a scale in between. Which of the two scales survives (if any)? How does the power spectrum look if you draw it before and after filtering?

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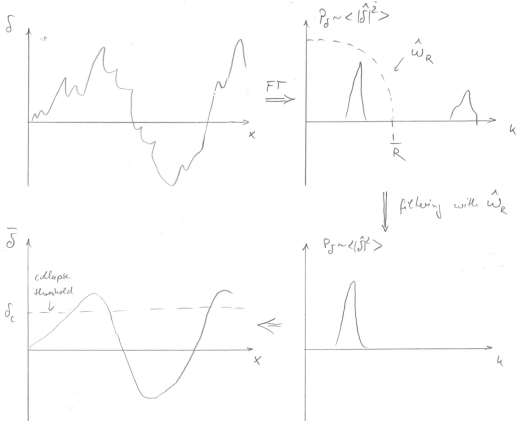
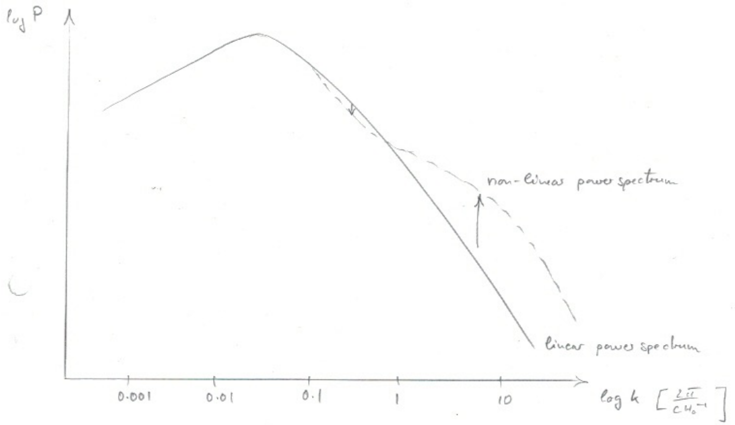


The matter power spectrum



- for a cold dark matter cosmology, the resulting power spectrum reads

$$P(k) \propto \begin{cases} k & (k < k_0) \\ k^{-3} & (k \gg k_0) \end{cases}$$

- here, $k_0 = 2\pi a_{\mathrm{eq}}/\lambda_0$ is the comoving wave number of the particle horizon at matter-radiation equality

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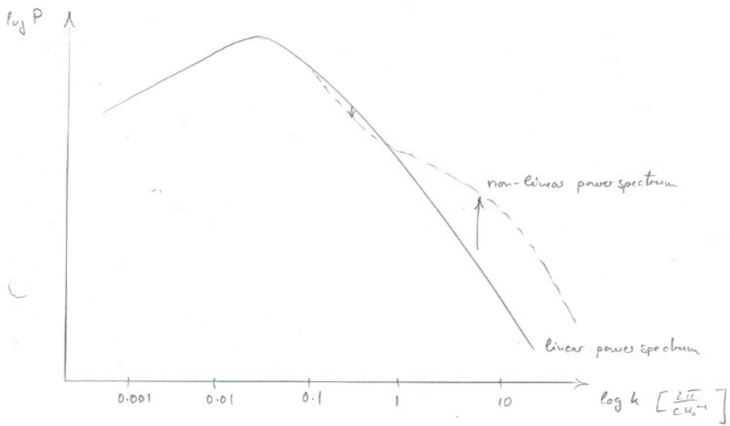

- non-linear structure formation causes more strongly enhanced density fluctuations on small scales
- development of a bump at large wave vectors (small spatial scales) in the non-linear matter power spectrum at the expense of intermediate scales

Hierarchical formation: when do clusters form?

Quantifying the variance of the matter density fluctuations

- We define the *non-linear mass* M_*, as the mass contained in a sphere of radius $R_* = 2\pi/k_*$ on which the variance becomes unity:

$$\sigma_*^2 = \int_0^{k_*} \frac{\mathrm{d}^3 k}{(2\pi)^3} P(k) \overset{!}{=} 1,$$

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- Masses and length scales are related by background density, $M \propto \bar{\rho} R^3 \propto k^{-3}$. For a fixed volume, density fluctuations are related to mass fluctuations in this volume, $\delta\rho \propto \delta M$. Normalized by the average density, we obtain $\delta \propto \delta M/M$.

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- Using the definition of the variance and the power spectrum $\sigma^2 \propto k^3 P(k) \propto \delta^2$, we get

$$\sigma^2 = \left\langle \left(\frac{\delta M}{M}\right)^2 \right\rangle = \left(\frac{M}{M_*}\right)^{-1-n/3} = \begin{cases} \left(\frac{M}{M_*}\right)^{-4/3} & \text{for } n = 1, \\ 1 & \text{for } n = -3. \end{cases} \quad (1)$$

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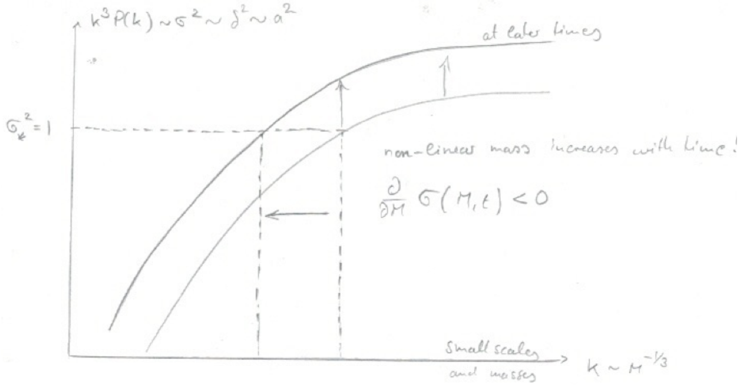
Hierarchical formation: when do clusters form?

Variance of the matter density fluctuations as a function of wave number, $\sigma^2(k)$



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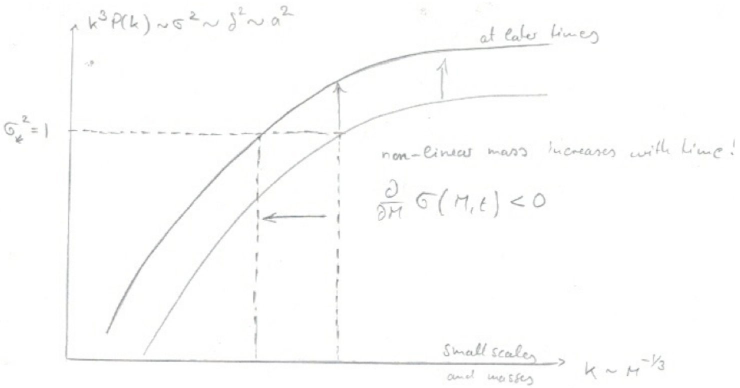

- more power on small scales, which first go non-linear ($\sigma^2 > 1$) and thus collapse first: dwarf galaxies form before large galaxies, which form before galaxy clusters

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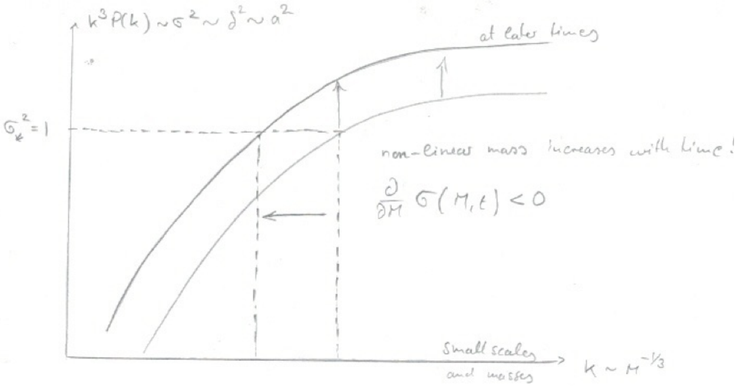

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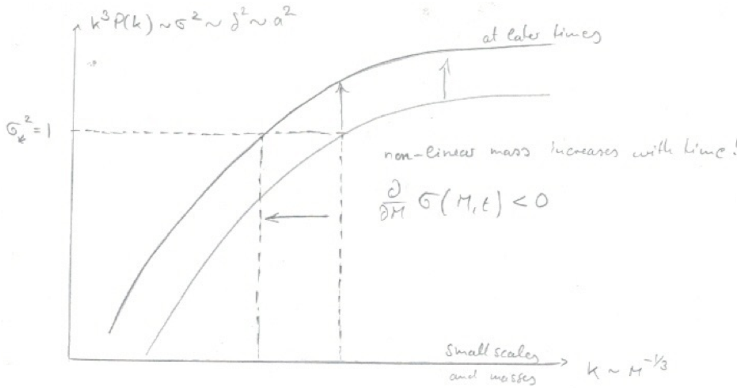

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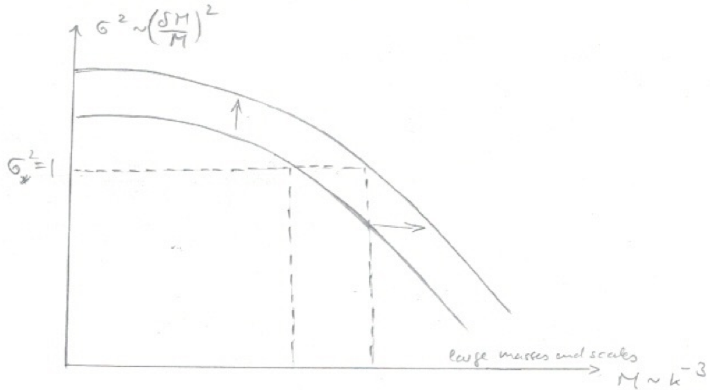
Variance of the matter density fluctuations as a function of collapsed mass, $\sigma^2(M)$



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- What physics would you have to change in such a universe?
  $\Rightarrow$ dark matter needs to be relativistic during freeze-out in the early universe
  $\Rightarrow$ free streaming of dark matter wipes out all fluctuations on scales below the mean free path $\lambda_{\mathrm{mfp}} \sim m_{\mathrm{DM}}/(\rho_{\mathrm{DM}}\sigma)$
  $\Rightarrow$ this is the case of "hot dark matter" which is ruled out by the observation that galaxies form before clusters

The initial power spectrum

Potential fluctuations make case for Harrison-Zel'dovich-Peebles spectrum

- Let's look at the fluctuations in the gravitational potential (at fixed volume),

$$\delta\Phi \sim \frac{GM}{R}\frac{\delta M}{M} \sim GM^{2/3}\bar{\rho}^{1/3}\frac{\delta M}{M}$$

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- Unless $\delta M/M \propto M^{-2/3}$, the potential fluctuations $\delta\Phi$ will diverge. Depending on the power-law index of $\delta M/M \propto M^{-\alpha}$, $\delta\Phi$ will diverge on large scales or masses (for $\alpha < 2/3$) or on small scales or masses (for $\alpha > 2/3$).

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- Hence, the most natural fluctuation spectrum is $\delta M/M \propto M^{-2/3}$, which avoids divergences.
- This can be related to a power spectrum in wave number space by considering $\delta\Phi \sim Gk\delta M$ and $M \propto R^3 \propto k^{-3}$ which yields (Eqn. 1)

$$\frac{\delta M}{M} \propto M^{-(n+3)/6} \quad \Rightarrow \quad \delta M \propto M^{-(n-3)/6} \propto k^{(n-3)/2},$$

  or

$$\delta\Phi \propto k^{(n-1)/2}.$$

  This shows that $n = 1$ is the characteristic spectral index that avoids any unphysical divergence and corresponds to the so-called Harrison-Zel'dovich-Peebles spectrum of initial fluctuations.

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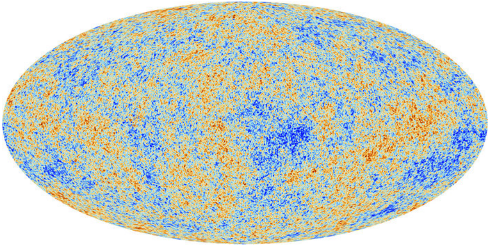
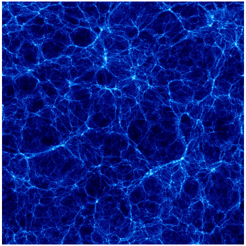
The CMB sky and the large-scale matter distribution





- **Left:** the cosmic microwave background fluctuations as observed by the Planck Collaboration (2013)
- **Right:** the cosmic web-like large-scale structure of the universe, dominated by dark matter

The CMB vs. the matter power spectrum – 1

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- **Matter power spectrum:** the density contrast δ is decomposed into plane waves, the orthonormal basis functions of a periodic box. The Fourier transform $\hat{\delta}$ of the density contrast δ is defined as

$$\delta(\boldsymbol{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3}\, \hat{\delta}(\boldsymbol{k}) \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} , \quad \hat{\delta}(\boldsymbol{k}) = \int \mathrm{d}^3 x\, \delta(\boldsymbol{x}) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} .$$

so that the power spectrum $P(k)$ is given by

$$\langle \hat{\delta}(\boldsymbol{k}) \hat{\delta}^*(\boldsymbol{k}') \rangle \equiv (2\pi)^3 P(k) \delta_{\mathrm{D}}(\boldsymbol{k} - \boldsymbol{k}') ,$$

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- **CMB power spectrum:** Fourier decomposition is not defined on the sphere. Instead, one has to project the temperature fluctuations onto another set of basis functions which are orthonormal on the sky. These are the spherical harmonic functions $Y_\ell^m(\boldsymbol{\theta})$. If $T(\boldsymbol{\theta})$ is the temperature at position $\boldsymbol{\theta}$ on the sky, it can be expanded into a series

$$T(\boldsymbol{\theta}) = \sum_{\ell m} a_{\ell m} Y_\ell^m(\boldsymbol{\theta})\,, \quad \text{where} \quad a_{\ell m} = \int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}\sin\theta\mathrm{d}\theta\, T(\theta,\phi)Y_\ell^m(\theta,\phi)\,,$$

  are the (generally complex) expansion coefficients and the power spectrum of the temperature map is defined by

$$C_\ell = \left\langle|a_{\ell m}|^2\right\rangle = \sum_{\mathrm{m}=-\ell}^{\ell}|a_{\ell m}|^2$$

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The CMB vs. the matter power spectrum – 2

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  Before the release of the CMB radiation, photons and baryonic matter were tightly coupled via Thompson scattering and the fluctuations in the CMB mirror those in the baryons.
- In the standard model of cosmology, fluctuations in DM and baryons have a common origin but fluctuations in the DM density are larger than that in the baryons because DM does not interact electromagnetically.
- Hence the matter density fluctuations (statistically quantified by the matter power spectrum) provide a much later, non-linear snapshot of cosmic structure and the temperature fluctuations observed in the CMB (and quantified in the CMB power spectrum) provide the initial conditions for structure formation. The amplitude of both scales with the matter density parameter Ω_m.

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Evolution of the matter power spectrum

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- The small baryonic component (by mass) oscillates with the photons before they decouple from the primordial plasma (release of the radiation that forms the CMB today): generation of small oscillations imprinted onto the linear matter power spectrum (with a wavelength that corresponds to the sound horizon at the time of decoupling); the same oscillations are seen in the CMB power spectrum.

Evolution of the matter power spectrum

- Cosmic inflation (or another process shortly after the Big Bang) sets the Harrison-Zel'dovich-Peebles spectrum, $P_i(k) \propto k^n$ with $n \approx 1$ (exact measurements indicate $n = 0.96$)
- Once cold dark matter decouples from the thermal heat bath (the "freeze-out") in the early universe, it attains a small velocity dispersion and wipes out structures on the smallest scales: generation of cutoff in the primordial power spectrum at scales of Earth-mass halos (uncertain by a factor of 10^6).
- Modes that enter the horizon during radiation domination are suppressed because the rapid cosmic expansion is faster in comparison to the gravitational collapse. They can only continue to grow after the universe transitions to matter domination: the power spectrum on intermediate to small scales is suppressed by k^4 so that the linear power spectrum scales as $P(k) \propto k$ (large scales) and $P(k) \propto k^{-3}$ (intermediate to small scales).
- The small baryonic component (by mass) oscillates with the photons before they decouple from the primordial plasma (release of the radiation that forms the CMB today): generation of small oscillations imprinted onto the linear matter power spectrum (with a wavelength that corresponds to the sound horizon at the time of decoupling); the same oscillations are seen in the CMB power spectrum.
- Smallest scales go first non-linear and collapse to small (dwarf-sized) dark matter halos: development of enhanced small-scale fluctuations and a bump in the non-linear power spectrum.

Non-linear structure formation: numerical simuations

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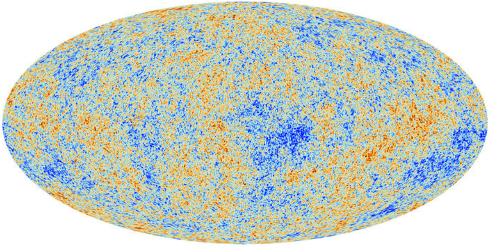
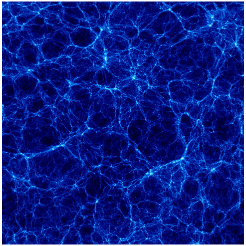
- Describe in your own words which algorithms are improving the scaling properties of numerical codes with the numbers of particles.
  1. direct summation,
  2. particle-mesh (PM) algorithm,
  3. particle-particle particle-mesh (P^3M) algorithm,
  4. tree algorithm,
  5. combination of tree-PM algorithm with direct summation on very small scales

The CMB sky and the large-scale matter distribution





- **Left:** the cosmic microwave background fluctuations as observed by the Planck Collaboration (2013)
- **Right:** the cosmic web-like large-scale structure of the universe, dominated by dark matter

AIP

Non-linear structure formation: density fluctuations

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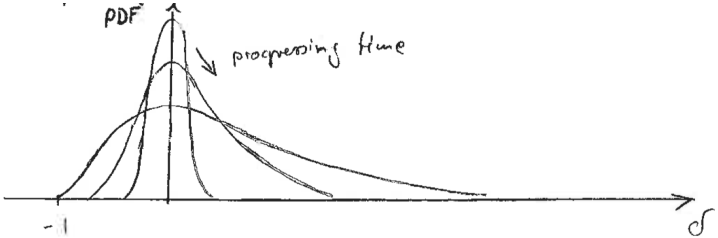
- What is the reason that the distribution becomes skewed at late times?

Non-linear structure formation: density fluctuations



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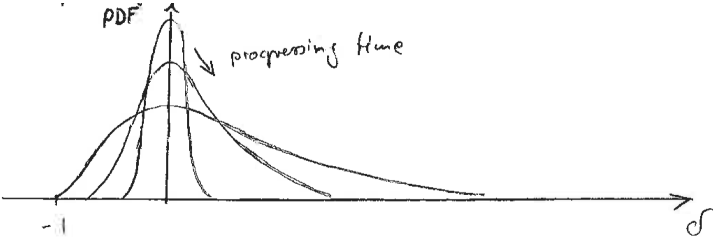

- Plot the probability distribution function of the density contrast at early and at late times after non-linear structure formation has already begun?
- What is the reason that the distribution becomes skewed at late times?
  $\Rightarrow$ The overdensity is defined as

$$\delta = \frac{\rho - \bar{\rho}}{\bar{\rho}} > -1$$

  and is not bound from above.
  $\Rightarrow$ It forms a log-normal distribution at late times!

- Movies of structure formation simulations
- Which properties that we talked about so far can you recognize in these simulations?
- Which things do you not understand or would like to know more about?

Spherical collapse: assumptions

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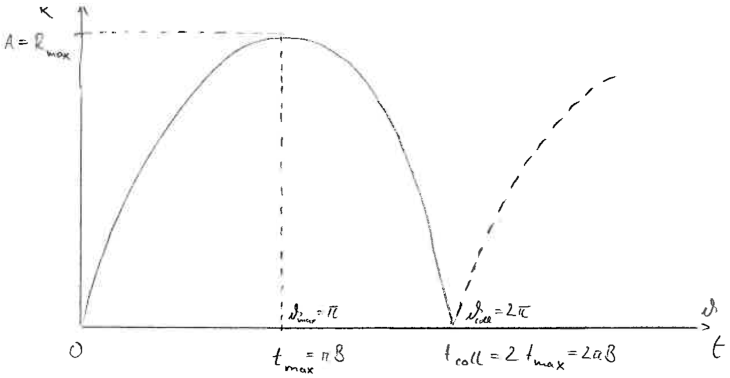
  3. for simplicity, $\Omega = \Omega_\mathrm{m} = 1$; this can be generalized to cases with $\Omega_\mathrm{m0} \neq 1$ and $\Omega_\Lambda \neq 1$.
- What is the benefit of doing this calculation if you have to assume these numbers of simplifications?

AIP

Spherical collapse: the cycloidal solution



- The solution is parametrized with θ and periodic beyond $\theta = 2\pi$.
- Is this completely unphysical? Why is the solution for $\theta > 2\pi$ not realized in nature?

Spherical collapse: overdensities

- The spherical collapse problem has the following parametric solution, which describes a cycloid,

$$R = A(1 - \cos\theta)\ , \quad \text{and} \quad t = B(\theta - \sin\theta).$$

AIP

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- The mean density inside the sphere is

$$\rho = \frac{M}{4\pi/3\, R^3} = \frac{3M}{4\pi A^3} \frac{1}{(1 - \cos\theta)^3} ,$$

while the mean density of the background universe with $\Omega_{m0} = 1$ is

$$\bar{\rho} = \frac{3H^2}{8\pi G} = \frac{1}{6\pi G t^2} = \frac{1}{6\pi G B^2} \frac{1}{(\theta - \sin\theta)^2} ,$$

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- The overdensity of the sphere (which is generally non-linear) can be obtained by combining these equations to yield

$$1 + \delta = \frac{\rho}{\bar{\rho}} = \frac{9}{2}\frac{(\theta - \sin\theta)^2}{(1-\cos\theta)^3}\ . \qquad (2)$$

Spherical collapse: turnaround

- There is an important distinction between (1) the real overdensity and (2) the overdensity *extrapolated* according to linear theory,

$$\delta_{\mathrm{lin}} = \delta_{\mathrm{i}} \left(\frac{t}{t_{\mathrm{i}}}\right)^{2/3} = \frac{3}{20}(6\pi)^{2/3} \left(\frac{t}{t_{\mathrm{ta}}}\right)^{2/3} \quad \text{for all } t.$$

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- The linear density at maximum expansion radius (i.e., at turnaround $t = t_{\rm ta}$) is

$$\delta_{\rm lin}(t_{\rm ta}) = \frac{3}{20}(6\pi)^{2/3} \approx 1.062$$

while the real (non-linear) overdensity at turnaround ($\theta = \pi$) is

$$1 + \delta(t_{\rm ta}) = \frac{9}{2}\frac{(\theta - \sin\theta)^2}{(1 - \cos\theta)^3} = \frac{9\pi^2}{16} \approx 5.55.$$

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- Why do perturbations collapse earlier if they are initially more over overdense?
  $\Rightarrow$ Because $t_\mathrm{c} \propto \delta_\mathrm{i}^{-3/2}$, the collapse time scales inversely with the initial overdensity.

AIP

Spherical collapse: final density – 1

- We assume that the *final* dark matter halo is in dynamical equilibrium and obeys the virial theorem

$$2K_\mathrm{f} + V_\mathrm{f} = 0\ ,$$

where K denotes the total kinetic energy in random motions, V is the total gravitational binding energy, and we neglected the surface pressure term due to further infalling material.

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- To calculate the total gravitational binding energy of a homogeneous sphere, we write down the masses of a shell and the sphere contained within it,

$$\mathrm{d}m_\mathrm{shell} = 4\pi r^2 \rho \mathrm{d}r \qquad \text{and} \qquad m_\mathrm{interior} = \frac{4}{3}\pi r^3 \rho.$$

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$$\mathrm{d}V_\mathrm{f} = -G\frac{m_\mathrm{interior}\mathrm{d}m_\mathrm{shell}}{r}.$$
- This can be integrated to obtain the total gravitational binding energy:
$$V_\mathrm{f} = -G\int_0^M \frac{4\pi r^3\rho}{3r}\mathrm{d}m_\mathrm{shell} = -G\frac{16}{3}\pi^2\rho^2\int_0^{R_\mathrm{f}} r^4\mathrm{d}r$$
$$= -G\frac{16}{15}\pi^2\rho^2 R_\mathrm{f}^5 = -\frac{3}{5}\frac{GM^2}{R_\mathrm{f}}.$$
- In the last step, we eliminated ρ by adopting the density of a homogeneous sphere, $\rho = M/[(4/3)\pi R_\mathrm{f}^3]$.

AIP

Spherical collapse: final density – 2

- Hence, we have the kinetic and gravitational binding energies:

$$K_\mathrm{f} = \frac{M}{2}\sigma_\mathrm{f}^2\,, \quad \text{and}$$
$$V_\mathrm{f} = -\frac{3}{5}\frac{GM^2}{R_\mathrm{f}}\,,$$

  where σ is the three-dimensional velocity dispersion and obtain the total energy

$$E_\mathrm{f} = K_\mathrm{f} + V_\mathrm{f} = \frac{1}{2}V_\mathrm{f} = -\frac{3}{10}\frac{GM^2}{R_\mathrm{f}}\,.$$

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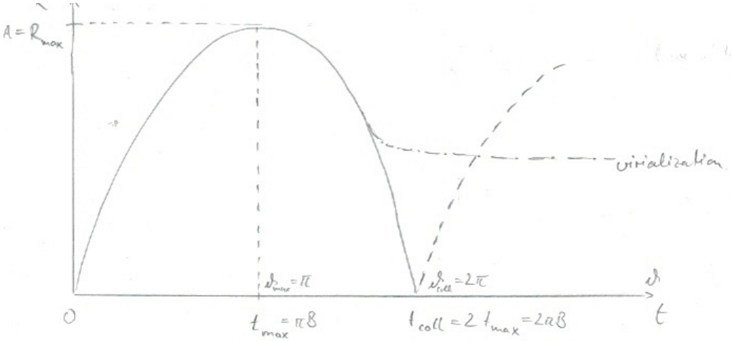
- At turn-around, the sphere is at rest, i.e., $K_\mathrm{ta} = 0$. The total energy at turn-around is

$$E_\mathrm{ta} = V_\mathrm{ta} = -\frac{3}{5}\frac{GM^2}{R_\mathrm{ta}}\,.$$

- Since dark matter is collisionless, the conservation of total energy during the collapse yields $E_\mathrm{f} = E_\mathrm{ta}$ and hence, $R_\mathrm{f} = R_\mathrm{ta}/2$.

Spherical collapse: picture of virialization



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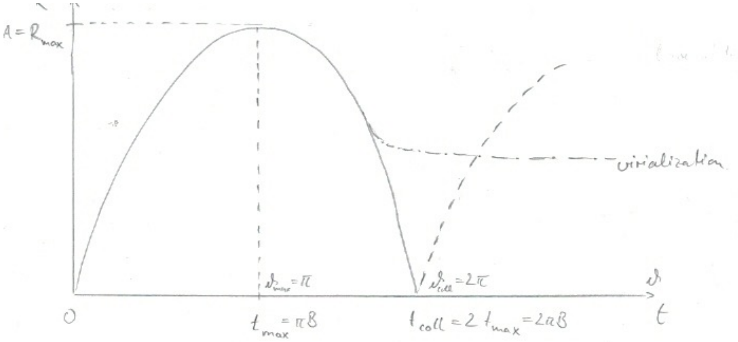

- Last week, I asked the question that, in the notes, I state that we typically find $\bar{\rho}_{\mathrm{cl}} \sim 10^3 \bar{\rho}_{\mathrm{m},0}$. Which processes determine this relation? What is the answer in the spherical collapse model?

Spherical collapse: virialization

- The final density is thus $\rho_\mathrm{f} = 8\rho(t_\mathrm{ta})$. Assuming that virialization happens at $t \approx t_\mathrm{c}$ and since $\bar{\rho} \propto t^{-2}$ and $t_\mathrm{c} = 2t_\mathrm{ta}$, the overdensity of the final halo is

$$1 + \delta_\mathrm{v} \equiv 1 + \delta_\mathrm{coll} = \frac{\rho_\mathrm{coll}}{\bar{\rho}\,(t_\mathrm{c}/t_\mathrm{ta})^{-2}} = 32\,[1 + \delta(t_\mathrm{ta})] = 18\pi^2 = 178\ ,$$

where $\rho(t_\mathrm{ta})/\bar{\rho} = 1 + \delta(t_\mathrm{ta})$ and we evaluated Eq. (2) at turn-around ($\theta = \pi$) so that $1 + \delta(t_\mathrm{ta}) = 9\pi^2/16$.

AIP

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- Hence, the final halo density is

$$\rho_\mathrm{f} = (1 + \delta_\mathrm{v})\bar{\rho}(t_\mathrm{c}) = 18\pi^2\bar{\rho}(t_\mathrm{c}).$$

Spherical collapse: characteristic overdensities

- We find values for the density contrast at collapse ($t = t_c = 2t_{\rm max}$) of

$$\delta_c \equiv \delta_{\rm lin}(t_c) = \frac{3}{20}(12\pi)^{2/3} \approx 1.686,$$
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  Explain the difference of these results that describe the same quantity at the same time.

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  Explain the difference of these results that describe the same quantity at the same time.
- We will later on use both results. Under which circumstances would you use the first and under which the second result?

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- Hierarchical structure formation: bottom-up growth of structure is consequence of shape of power spectrum/variance of density fluctuations

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  * need of numerical simulations because of non-linearity and high dimensionality
  * clever algorithms to improve scaling with number of particles
- Spherical Collapse:
  * relates time (or redshift) at which the object collapses to its initial (linear) overdensity
  * it maps the collapse time (redshift) to the final density of dark matter haloes that formed by collapse

AIP