The Physics of Galaxy Clusters
4th Lecture

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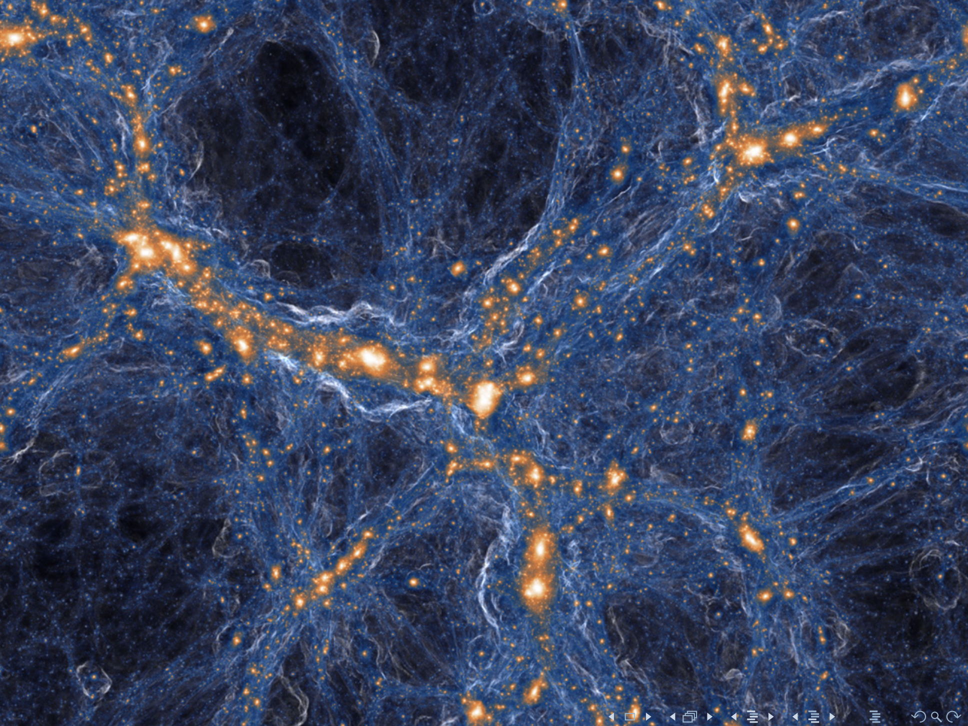
Lectures in the International Astrophysics Masters Program at Potsdam University



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$\Rightarrow$ Today's topic: we will learn how to count clusters and how clusters are built up.

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$$\sigma^2_{R_*} = 4\pi \int_0^\infty \frac{k^2 {\rm d}k}{(2\pi)^3} P(k) \hat{W}^2_{R_*}(k) = \delta_{\rm c}^2\,,$$

where $\delta_{\rm c} = 1.686$. Hence, if the density field is smoothed on the scale R_* then the variance of the resulting field is just equal to the collapse threshold.

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- This means that halos with mass M_* are forming today (or at the redshift that the analysis is performed at). Halos with $M < M_*$ have been abundantly forming in the past because their variance is larger than the collapse threshold and halos with $M > M_*$ are extremely rare (they must be large excursions) if they can form at all $\Rightarrow$ see "Simplified Form of the Mass Function"

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- Since the density fluctuations arise from superpositions of enormous numbers of statistically independent vacuum fluctuations of the inflaton field, they are expected to be Gaussian because of the central limit theorem.

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The derivation of the halo mass function – 1

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- We can picture this the following way: imagine a density field in two spatial dimensions that is visualized as a landscape with mountains and valleys.
- If we fill in water into this landscape up to the point, where the surface equals the collapse threshold δ_c, then the islands denominate the regions that will collapse into halos. Hence, the probability of collapsing into such a halo equals the likelihood of finding the islands, i.e., it is the fraction of surface area covered by islands.

The derivation of the halo mass function – 2

- The probability of finding a filtered density contrast $\bar{\delta}(\boldsymbol{x})$ at $\boldsymbol{x}$ is

$$p(\bar{\delta}, a) = \frac{1}{\sqrt{2\pi\sigma_R^2(a)}} \exp\left[-\frac{\bar{\delta}^2(\boldsymbol{x})}{2\sigma_R^2(a)}\right] ,$$

where the variance σ will depend on time or equivalently on the scale factor a through the linear growth factor, $\sigma_R(a) = \sigma_R D_+(a)$ ($D_+(a) = a$ in the Einstein-de-Sitter model with $\Omega_\mathrm{m}(a) = 1$).

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- We adopt the substitution

$$x = \frac{1}{\sqrt{2}} \frac{\bar{\delta}}{\sigma_R(a)} \quad \text{and} \quad \mathrm{d}x = \frac{1}{\sqrt{2}\sigma_R(a)} \mathrm{d}\bar{\delta}$$

  and obtain

$$F(M, a) = \frac{1}{2} \frac{2}{\sqrt{\pi}} \int_{\delta_{\mathrm{c}}/[\sqrt{2}\sigma_R(a)]}^{\infty} \mathrm{d}x e^{-x^2} = \frac{1}{2} \mathrm{erfc}\left(\frac{\delta_{\mathrm{c}}}{\sqrt{2}\sigma_R(a)}\right) ,$$

  where $\mathrm{erfc}(x)$ is the complementary error function.

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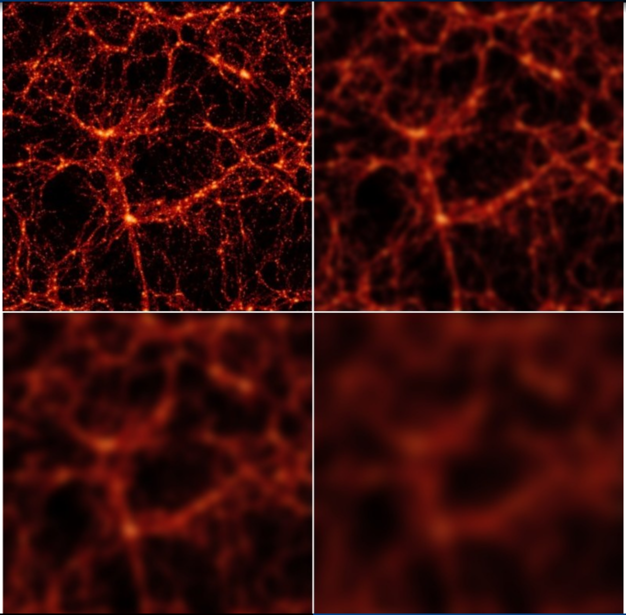
- We calculate the normalization of the Press-Schechter mass function:

$$\begin{aligned}\int_0^\infty \left|\frac{\partial F(M)}{\partial M}\right| \mathrm{d}M &= \frac{\delta_\mathrm{c}}{\sqrt{2\pi}D_+(a)} \int_0^\infty \frac{\mathrm{d}\sigma_R}{\sigma_R^2} \exp\left(-\frac{\delta_\mathrm{c}^2}{2\sigma_R^2 D_+^2(a)}\right) \\ &= \frac{\delta_\mathrm{c}}{\sqrt{2\pi}D_+(a)} \int_0^\infty \mathrm{d}x \exp\left(-\frac{\delta_\mathrm{c}^2}{2D_+^2(a)}\,x^2\right) \\ &= \frac{\delta_\mathrm{c}}{\sqrt{2\pi}D_+(a)} \frac{\sqrt{2\pi}}{2} \sqrt{\frac{D_+^2(a)}{\delta_\mathrm{c}^2}} = \frac{1}{2}\end{aligned}$$

where we adopted the substitutions $\sigma_R^{-1} = x$ so that $\mathrm{d}x = -\mathrm{d}\sigma_R/\sigma_R^2$.

Halo formation as a random walk – 1

Progressive smoothing of the density field



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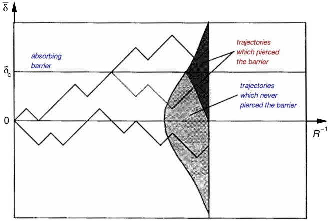

- Consider halo formation as a random walk $\Rightarrow$ correct normalisation of the Press-Schechter mass function
- Given the density-contrast field δ, a large sphere is centred on some point $\boldsymbol{x}$ and its radius gradually shrunk. For each radius R of the sphere, the density contrast $\bar{\delta}$ averaged within R is measured and monitored as a function of R.

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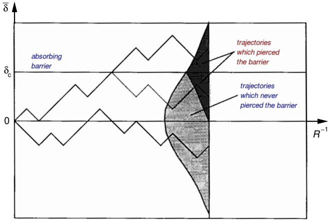

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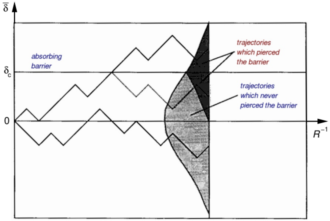

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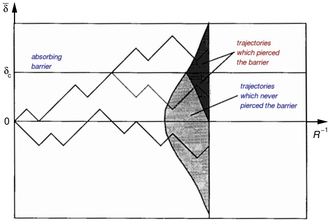
- By choosing a window function W_R whose Fourier transform has a sharp cut-off in k space, $\bar{\delta}$ will undergo a random walk because decreasing R corresponds to adding shells in k space which are independent of the modes which are already included.
- $\bar{\delta}(\boldsymbol{x})$ is thus following a random trajectory. A halo is expected to be formed at $\boldsymbol{x}$ if $\bar{\delta}(\boldsymbol{x})$ reaches δ_c for some radius R.
- If $\bar{\delta}(\boldsymbol{x}) < \delta_c$ for some radius, it may well exceed δ_c for a smaller radius. Or, if $\bar{\delta}(\boldsymbol{x}) \geq \delta_c$ for some radius, it may well drop below δ_c for a smaller radius.

Halo formation as a random walk – 3



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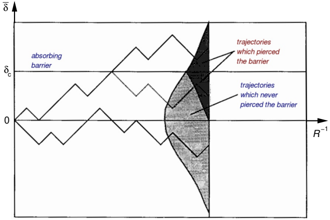

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- We introduce an *absorbing barrier* at δ_c such that points $\boldsymbol{x}$ with trajectories $\bar{\delta}(\boldsymbol{x})$ vs. R which hit the barrier for some R are considered to be part of halos.
- A trajectory meeting the boundary (for the first time) at radius R' has equal probability for moving above or below at smaller R. For any trajectory continuing above the boundary (and thus being part of a halo), there is a mirror trajectory that falls back below the boundary at $R < R'$.

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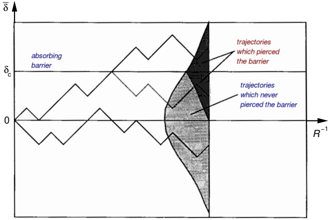

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- However, as this latter trajectory has reached the boundary at R' it will be part of a halo with mass $\sim M(R')$. Thus, for each trajectory piercing the barrier for the first time at R' and reaching $\bar\delta(\boldsymbol{x}) > \delta_c$ at R there is another trajectory, which reached the barrier at R' but has $\bar\delta(\boldsymbol{x}) < \delta_c$ at R. Thus, the integral in Eqn. (2.78) recovers only half of the points that end up in a halo of mass $\geq M(R)$.

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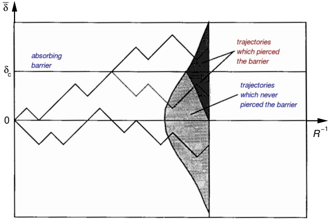

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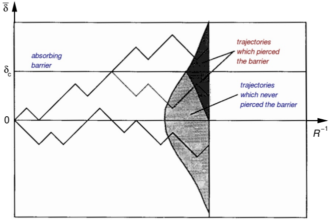
- It missed the dark grey halo population below the barrier that have pierced (or touched) the barrier at some smoothing radius $R' > R$.



- Thus, the probability for reaching a point $\bar\delta < \delta_\mathrm{c}$ along trajectories exclusively below the barrier after piercing the absorbing barrier is the probability for reaching it along *any* trajectory, minus the probability for reaching its mirror point $\delta_\mathrm{c} + (\delta_\mathrm{c} - \bar\delta) = 2\delta_\mathrm{c} - \bar\delta$ along trajectories that continue above the barrier,

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where σ_R is the variance of $\bar\delta$ on the scale R, as before.

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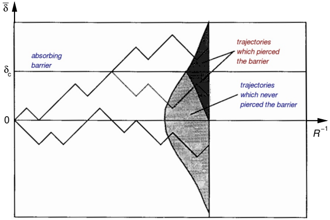

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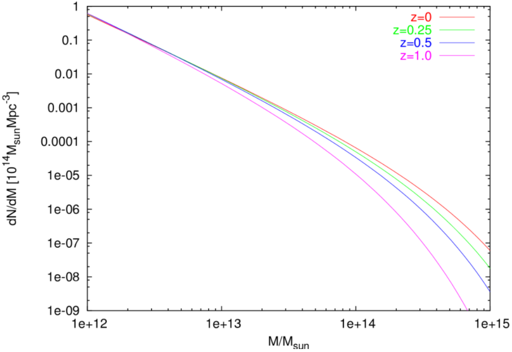
- This equation is the probability distribution for the averaged density contrast to fall within $[\bar\delta, \bar\delta + \mathrm{d}\bar\delta]$ and *not* to exceed δ_c when averaged on *any* scale. The probability for $\bar\delta$ to *exceed* δ_c on some scale is thus

$$1 - P_s = 1 - \int_{-\infty}^{\delta_c} \mathrm{d}\bar\delta\, p_s(\bar\delta) = \mathrm{erfc}\left(\frac{\delta_c}{\sqrt{2}\sigma_R}\right) ,$$

without the factor $1/2$ in the previous derivation.

Simplified form of the mass function



- For a power-law power spectrum with index n, $P_\delta(k) = Ak^n$, the Press-Schechter mass function is given by ($m = M/M_*$)

$$f(m,a)\mathrm{d}m \equiv \frac{\mathrm{d}N(m,a)}{\mathrm{d}m}\mathrm{d}m \propto m^{\alpha-2}\exp\left(-m^{2\alpha}\right)\mathrm{d}m,$$

where we defined $\alpha = 1/2 + n/6$ so that $\alpha = 0$ for $n = -3$.

- At small halo masses there is roughly an equal mass per log bin in halo mass,

$$m\,\mathrm{d}N/\mathrm{d}\log m = m^2\mathrm{d}N/\mathrm{d}m \approx \text{const.}$$

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- This can happen through the ejection of a body by means of three-body encounters. As the halo becomes more tightly bound, this in turn increases its energy loss because the dynamical timescale is reduced by this contraction: as the system ejects (kinetic) energy, it increases its potential energy which means that a self-gravitating system has a negative heat capacity.

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- Assume that you have a globular cluster of 10^6 stars and size 10 pc. Explain what happens to the system when you eject one star after each other. What is the theoretical end state?

Halo density profiles

- **General remarks.** Scrutinize the statement that *a self-gravitating system of particles does not have an equilibrium state*.
- The virial theorem demands that its total energy ($E = K + V$) is minus half its potential energy (V),

$$2K + V = E + K = 0 \quad \Rightarrow \quad K = -E = -\frac{V}{2}.$$

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- Assume that you have a globular cluster of 10^6 stars and size 10 pc. Explain what happens to the system when you eject one star after each other. What is the theoretical end state?
- A binary star, but this state is never reached because of the slow timescales (dynamical friction).

Singular isothermal sphere

- Show explicitly that a power-law ansatz in $\rho(r)$ yields the expression for the singular isothermal sphere,

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\ln\rho}{\mathrm{d}r}\right) = -\frac{4\pi G m}{k_\mathrm{B} T} r^2 \rho \, .$$

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- This has the solution

$$\alpha = 2 \quad\text{and}\quad C = \frac{k_\mathrm{B}T}{2\pi Gm}$$

or

$$\rho(r) = \frac{\sigma^2}{2\pi Gr^2} \quad\text{with}\quad \sigma^2 \equiv \frac{k_\mathrm{B}T}{m}\,.$$

AIP

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- Simulations of collapsing collisionless dark matter show a dark matter density profile that differs from the isothermal sphere. Instead, the profile is better fit by Navarro-Frenk-White (NFW) density profile.

AIP

Navarro-Frenk-White density profile

What is the logarithmic slope, $\mathrm{d}\log\rho/\mathrm{d}\log r$, of the NFW profile at the center, the scale radius r_s and at large radii?

$$\rho(r) = \frac{\rho_\mathrm{s}}{x(1+x)^2}\,, \quad x \equiv \frac{r}{r_\mathrm{s}}$$

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- $x \gg 1$: $\rho(r) = \rho_\mathrm{s} x^{-3}$ and the logarithmic slope is $\mathrm{d}\log\rho/\mathrm{d}\log x = -3$

AIP

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- Let's derive the mass profile $M(< r)$ of the NFW profile by using the identity $x/(1+x)^2 \equiv (1+x)^{-1} - (1+x)^{-2}$.

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- Compare the different definitions for halo mass, M_{200}, $M_{200\mathrm{m}}$, and M_{500} and order them by increasing size,

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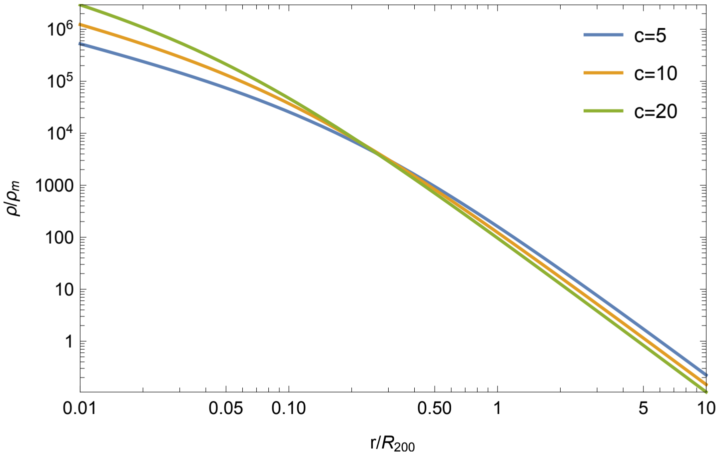
$$M_{500} < M_{200} < M_{200\mathrm{m}}.$$

Navarro-Frenk-White density profile

- Sketch qualitatively the scaled NFW density profiles $\log(\rho/\rho_{200})$ for two different halos which only differ by their concentration parameters. What do you observe? Which halo is on average more massive?

Navarro-Frenk-White density profile



- Sketch qualitatively the scaled NFW density profiles $\log(\rho/\rho_{200})$ for two different halos which only differ by their concentration parameters. What do you observe? Which halo is on average more massive?
- More concentrated halos have a larger density at small radii and are on average less massive because of the concentration-mass relation, $c_{200} \propto M_{200}^{-0.1}$.

⇒ We learned how to count clusters and how clusters are built up:

- The Press-Schechter mass function
  * probability of high-density excursions in the filtered density contrast on scale R = fraction of volume filled with halos of mass M
  * correct normalization: considering halo formation as a random walk and identify collapse threshold δ_c as absorbing barrier

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- The Press-Schechter mass function
  - probability of high-density excursions in the filtered density contrast on scale R = fraction of volume filled with halos of mass M
  - correct normalization: considering halo formation as a random walk and identify collapse threshold δ_c as absorbing barrier
- Halo density profiles:
  - self-gravitating systems have a negative heat capacity: once they cool they become hotter
  - singular/cored isothermal sphere is a simple model for spherically-symmetric, self-gravitating systems of single-temperature particles
  - Navarro-Frenk-White (NFW) density profile is a two-parameter model (normalization + scale radius or halo mass + concentration) measured in dark-matter simulations