The Physics of Galaxy Clusters
5th Lecture

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Lectures in the International Astrophysics Masters Program at Potsdam University

$\Rightarrow$ We learned how to count clusters and how clusters are built up:

- **The Press-Schechter mass function**
  * probability of high-density excursions in the filtered density contrast on scale
    R = fraction of volume filled with halos of mass M
  * correct normalization: considering halo formation as a random walk and
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- **Halo density profiles**
  * self-gravitating systems have a negative heat capacity: once they cool they become hotter
  * singular/cored isothermal sphere is a simple model for spherically-symmetric, self-gravitating systems of single-temperature particles
  * Navarro-Frenk-White (NFW) density profile is a two-parameter model normalization + scale radius or halo mass + concentration) measured in dark-matter simulations

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$\Rightarrow$ today: adiabatic processes and conservation laws

Specific heats – 1

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- The specific heat at constant volume is defined as

$$c_V \equiv \left(\frac{\partial q}{\partial T} \right)_V.$$

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At constant volume, the internal energy can only be changed by adding or releasing heat, $d\epsilon = dq$.

- If ϵ depends only on temperature (and not density), $\epsilon(\tilde{V}, T) = \epsilon(T)$, then

$$c_V \equiv \left(\frac{\partial q}{\partial T} \right)_V = \left(\frac{\partial \epsilon}{\partial T} \right)_V = \frac{\partial \epsilon}{\partial T}$$

implying

$$dq = c_V dT + Pd\tilde{V}.$$

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Specific heats – 2

● First law of thermodynamics with c_V:

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● The pressure of a gas of particles with mean mass $\bar{m}$ is given by

$$P\tilde{V} = \frac{k_\mathrm{B} T}{\bar{m}} \implies P\mathrm{d}\tilde{V} = \frac{k_\mathrm{B}}{\bar{m}}\mathrm{d}T.$$

Using $\mathrm{d}q = c_V \mathrm{d}T + P\mathrm{d}\tilde{V}$, the specific heat at constant pressure is

$$c_P \equiv \left(\frac{\partial q}{\partial T}\right)_P = c_V + P\frac{\mathrm{d}\tilde{V}}{\mathrm{d}T} = c_V + \frac{k_\mathrm{B}}{\bar{m}}.$$

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- Why does changing the temperature at constant pressure require more heat than at constant volume?
  $\Rightarrow$ because some of the energy goes into $P\mathrm{d}\tilde{V}$ work

What is the adiabatic index for a diatomic gas that has five degrees of freedom (3 translational and 2 rotational)?

Adiabatic index

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- The specific energy of a gas with f_{DOF} number of degrees of freedom is

$$\epsilon = \frac{f_{DOF}}{2} \frac{k_B T}{\bar{m}} \implies c_V = \left(\frac{\partial \epsilon}{\partial T} \right)_V = \frac{f_{DOF}}{2} \frac{k_B}{\bar{m}}$$

$$\implies c_P = \frac{f_{DOF} + 2}{2} \frac{k_B}{\bar{m}}$$

$$\implies \gamma \equiv \frac{c_P}{c_V} = \frac{f_{DOF} + 2}{f_{DOF}}$$

Hence, $\gamma = 7/5$ for a diatomic gas with $f_{DOF} = 5$.

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  Hence, $\gamma = 7/5$ for a diatomic gas with $f_{DOF} = 5$.

- In general, the equation of state for an ideal gas is given by

$$\epsilon = \frac{f_{DOF}}{2} \frac{k_B T}{\bar{m}} = \frac{1}{\gamma - 1} \frac{k_B T}{\bar{m}}$$

- Check out Appendix A.1 on "Equation of State and Mean Molecular Weight", which introduces the mean molecular weight $\mu = \bar{m}/m_H$ to relate the mean particle mass $\bar{m}$ and the mass of a hydrogen atom m_H.

AIP

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$$d\epsilon = \frac{1}{\gamma - 1} \left(\frac{dP}{\rho} - \frac{P}{\rho^2} d\rho \right).$$

- For adiabatic ($dq = ds = 0$) changes, we can combine this with the first law of thermodynamics ($\tilde{V} = \rho^{-1}$)

$$d\epsilon = -P d\tilde{V} = \frac{P}{\rho^2} d\rho$$

  and find (after multiplying with ρ/P)

$$\frac{1}{\gamma - 1} \left(\frac{dP}{P} - \frac{d\rho}{\rho} \right) = \frac{d\rho}{\rho},$$

  implying

$$\frac{dP}{P} = \gamma \frac{d\rho}{\rho} \implies P = P_0 \left(\frac{\rho}{\rho_0} \right)^{\gamma} \equiv K \rho^{\gamma}.$$

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K &= \frac{P}{\rho^\gamma} = \frac{k_B T}{\bar{m} \rho^{\gamma-1}}, \text{ and} \\
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- K_e which can be conveniently computed with the X-ray observables $k_B T_e$ and n_e and has typical values of

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- For a single species gas, we have

$$
s = c_V \ln(P\rho^{-\gamma}) + \text{const.} = \frac{k_B}{(\gamma-1)\bar{m}} \ln K + \text{const.}
$$

or

$$
s = c_V \ln \left(\frac{K}{K_0} \right) \quad \Longleftrightarrow \quad K = K_0 \exp \left(\frac{s}{c_V} \right).
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Fluid description

When can a system be well described as a fluid? Is this justified in a low-mass and a high-mass cluster (with $k_B T = 1$ keV and 10 keV) on scales of a few kpc? What does this imply for modelling clusters?

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- A system can be well described as a fluid, if the particle mean free path is much shorter than the characteristic system size, $\lambda_{mfp} \ll L$. The electron mean free path is given by

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\begin{aligned}
\lambda_{mfp} &= \frac{1}{n\sigma \ln \Lambda} \sim \frac{1}{n\pi r_e^2 \ln \Lambda} \sim \frac{1}{\pi n \ln \Lambda} \left(\frac{k_B T_e}{Ze^2} \right)^2 \\
&\sim 12.6 \left(\frac{n}{10^{-3} \text{ cm}^{-3}} \right)^{-1} \left(\frac{k_B T_e}{10 \text{ keV}} \right)^2 \text{ kpc},
\end{aligned}
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where $\ln \Lambda \approx 40$ is the Coulomb logarithm. Equivalently, we get $\lambda_{mfp} = 0.14$ kpc for $k_B T_e = 1$ keV (which implies $\ln \Lambda \approx 36$). Since $\lambda_{mfp} < L \sim$ a few kpc, the fluid description is justified for a small cluster according to standard text book knowledge. However, in a massive cluster $\lambda_{mfp} > L \sim$ a few kpc and we would have to use kinetic plasma theory.

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- In particular on small scales and in cluster outskirts, the fluid description is not justified and would have to be supplemented by kinetic plasma physics. Moreover, plasma physics on scales $\lambda_{mfp} < 0.1 - 10$ kpc could modify the effective fluid prescription and could feedback on global cluster scales $\Rightarrow$ subject to active research!

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Conservation laws – 1

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- The Boltzmann equation describes the evolution of the phase space distribution function $f(\boldsymbol{x}, \boldsymbol{u}, t)$ and is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\boldsymbol{x}, \boldsymbol{u}, t) = \frac{\partial f}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} f + \boldsymbol{g} \cdot \boldsymbol{\nabla}_u f = \left. \frac{\mathrm{d}f}{\mathrm{d}t} \right|_c.$$

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- The term $\mathrm{d}f/\mathrm{d}t|_c$ represents discontinuous motions of particles through phase space as a result of collisions. While collisions happen at a fixed point in space, they can instantaneously change particle velocities and thus cause particles to jump in phase space.

- We take mass, momentum, and energy moments of this equation to derive the conservation laws. The right-hand side vanishes each time because of local . . .

  1. **mass conservation:** collisions do not create or destroy particles at a fixed position, they can only discontinuously shift them in velocity space.

  2. **momentum conservation:** collisions do not create or destroy momentum at a fixed position, the shift in momentum space is symmetric with respect to the center of momentum (neglecting internal degrees of freedom such as atomic or nuclear transitions during the interaction).

  3. **energy conservation:** collisions do not create or destroy energy at a fixed position (again neglecting internal degrees of freedom).

Show explicitly that the Navier-Stokes equation (Eq. 3.43) conserves momentum.

$$\frac{\partial}{\partial t}(\rho \boldsymbol{v}) + \boldsymbol{\nabla} \cdot \left(\rho \boldsymbol{v} \boldsymbol{v}^{\mathsf{T}} + P\bar{\boldsymbol{1}} - \bar{\bar{\boldsymbol{\Pi}}} \right) = \rho \boldsymbol{g}.$$

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- Integrating the Navier-Stokes equation over configuration space yields

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\begin{array}{rcl}
\displaystyle \frac{\partial}{\partial t} \int_{\Omega} \rho \boldsymbol{v} \mathrm{d}^3 x + \int_{\Omega} \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v} \boldsymbol{v}^{\mathsf{T}} + P\bar{\mathbf{1}} - \bar{\mathbf{\Pi}}) \mathrm{d}^3 x & = & \displaystyle \int_{\Omega} \rho \boldsymbol{g} \mathrm{d}^3 x, \\[2ex]
\displaystyle \frac{\partial \boldsymbol{p}_{\mathrm{tot}}}{\partial t} + \lim_{\partial \Omega \to \infty} \int_{\partial \Omega} [(\rho \boldsymbol{v} \boldsymbol{v}^{\mathsf{T}} + P\bar{\mathbf{1}} - \bar{\mathbf{\Pi}}) \cdot \boldsymbol{n}] \mathrm{d}^2 A & = & \displaystyle \frac{\mathrm{d} \boldsymbol{p}_{\mathrm{tot}}}{\mathrm{d} t} = \int_{\Omega} \rho \boldsymbol{g} \mathrm{d}^3 x.
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  In the last line, we exchanged the total for the partial time derivatives since $\boldsymbol{p}_{\text{tot}}$ depends neither on position nor on velocity.

- This demonstrates explicitly that the total momentum, $\boldsymbol{p}_{\text{tot}} = \int \rho \boldsymbol{v} \mathrm{d}^3 x$ is conserved in the absence of an external force field $\boldsymbol{g}$ that acts as a source of momentum.

AIP

Why is the anisotropic viscous stress tensor, $\bar{\bar{\Pi}}$, traceless and symmetric?

- The anisotropic viscous stress tensor, $\bar{\bar{\Pi}}$, is defined as

$$P \equiv \frac{1}{3}\rho\langle|\boldsymbol{w}|^2\rangle,$$
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● It is a traceless tensor because

$$\text{tr}\left(\bar{\bar{\Pi}}\right) \equiv 3P - \rho\langle|\boldsymbol{w}|^2\rangle = \rho\langle|\boldsymbol{w}|^2\rangle - \rho\langle|\boldsymbol{w}|^2\rangle = 0.$$

- The momentum equation can be written as

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = \boldsymbol{g} - \frac{1}{\rho}\boldsymbol{\nabla}P + \frac{1}{\rho}\boldsymbol{\nabla} \cdot \bar{\boldsymbol{\Pi}}.$$

- Explain, what viscosity does to shearing and interpenetration motion.

Anisotropic viscous stress tensor – 2

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- To see this, we adopt an ansatz for the viscous stress tensor and assume a "Newtonian fluid", i.e., we assume that Π_{ij} is linearly proportional to the velocity gradient, $\partial v_i/\partial x_j$. The most general symmetric tensor that is linear in $\partial v_i/\partial x_j$ is

$$\begin{aligned} \Pi_{ij} &= \eta D_{ij} + \xi \delta_{ij}(\boldsymbol{\nabla} \cdot \boldsymbol{v}), \text{ where} \\ D_{ij} &= \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta_{ij}(\boldsymbol{\nabla} \cdot \boldsymbol{v}) \end{aligned}$$

is the deformation tensor that vanishes for uniform expansion or contraction. η and ξ are the coefficients of shear and bulk viscosity, respectively and have units of g cm^{-1} s^{-1}.

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- *Bonus:* derive the energy conservation equation.

AIP

Conductive heat flux and entropy equation

Name and explain the condition for a non-vanishing conductive heat flux $\boldsymbol{Q}$.

- The conductive heat flux $\boldsymbol{Q}$ is defined as:

$$\boldsymbol{Q} \equiv \frac{1}{2} \rho \left\langle \boldsymbol{w} |\boldsymbol{w}|^2 \right\rangle .$$

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- $-P\boldsymbol{\nabla} \cdot \boldsymbol{v}$ term has to conserve entropy and accounts for adiabatic changes: converging flow has $\boldsymbol{\nabla} \cdot \boldsymbol{v} < 0 \Rightarrow$ energy is increased (adiabatic heating)

AIP

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- Each of these equations is a continuity law, one for the mass, one for the momentum, and one for the total energy. The equations hence form a set of hyperbolic conservation laws (because of the divergence operator on the LHS). In the form given above, they do not form a complete set because these are 5 equations for 7 unknowns: ρ, ϵ, P, $\boldsymbol{v}$, Φ.

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- Hence, we need two further expressions that relate the pressure P and the gravitational potential Φ to the other thermodynamic variables.

System of hydrodynamic equations – 2

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- The equations relating P and Φ to the other variables are the equation of state and Poisson's equation. For an ideal gas, the equation of state reads

$$P = (\gamma - 1)\rho \epsilon$$

where $\gamma = c_p/c_v$ the ratio of specific heats. For a mono-atomic, non-relativistic gas, we have $\gamma = 5/3$.

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- The gravitational potential obeys Poisson's equation:

$$
\boldsymbol{\nabla}^2 \Phi = 4 \pi G \rho
$$

These 7 equations close the system of equations and make it well-posed.

AIP

The NFW density profile diverges in the center, i.e., for $x \to 0$. Gas filled in the halo's gravitational potential Φ satisfies hydrostatic Euler's equation

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- The ideal gas law is

$$P = \frac{\rho_{\text{gas}} k_{\text{B}} T}{\bar{m}} \quad \text{and we assume } k_{\text{B}} T = \text{const.}$$

- Substituting into Eq. (1) we get

$$\frac{\nabla P}{\rho_{\text{gas}}} = \frac{k_{\text{B}} T}{\bar{m}} \nabla \ln \rho_{\text{gas}} = -\nabla \Phi \qquad \text{since } \nabla T = 0$$

$$\Rightarrow \ln \rho_{\text{gas}} = -\frac{\bar{m} \Phi}{k_{\text{B}} T} + C \qquad \text{and } C = \text{const.}$$

$$\Rightarrow \rho_{\text{gas}} = e^{C} \exp\left(-\frac{\bar{m} \Phi}{k_{\text{B}} T}\right) = A \exp\left(-\frac{\bar{m} \Phi}{k_{\text{B}} T}\right)$$

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$$\Phi(r) = -\frac{GM_{\text{s}}}{r_{\text{s}}} \frac{\ln(1+x)}{x} \quad \text{with} \quad x = \frac{r}{r_{\text{s}}} \quad \text{and} \quad M_{\text{s}} = 4\pi r_{\text{s}}^3 \rho_{\text{s}}.$$

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- Using the virial theorem ($E_{pot} = -2E_{kin}$), we get

$$-\bar{m}\Phi_0 = 2 \times \frac{3}{2}k_B T$$

$$\Rightarrow \frac{\bar{m}\Phi_0}{k_B T} = -3$$

$$\Rightarrow -\frac{\bar{m}\Phi}{k_B T} = -\frac{\bar{m}\Phi_0}{k_B T}\frac{\ln(1+x)}{x} = 3\frac{\ln(1+x)}{x}$$

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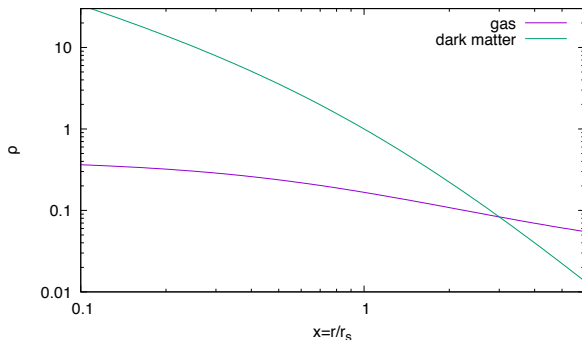
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- Differences are due to gas pressure of hot gas instead of non-interacting cold dark matter. What about large radii?

What happens to the gas-to-dark matter mass density ratio ρ_{gas}/ρ at large radii? Is this a realistic behavior and if not, what would have to be changed in the model to make it more realistic?

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● We consider the limit $x \to \infty$:

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instead of approaching the universal baryon fraction $\ln f_{\text{b}} \quad \Rightarrow \quad$ what is wrong?

● Our assumption $k_{\text{B}} T = $ const. is unrealistic. Including $\nabla T \neq \mathbf{0}$ in Euler's equation will change the gas density profile so that the constraint $\rho_{\text{gas}}/\rho \to f_{\text{b}}$ for $x \gg 1$ is fulfilled.

AIP

$\Rightarrow$ We studied adiabatic processes, conservation laws and filled gas into dark matter potentials:

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The Physics of Galaxy Clusters

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- Dark Matter halo profiles
  * in the exercises, you played with the NFW profile and derived the corresponding potential
  * we filled gas in the NFW potential and derived the density profile assuming an isothermal gas in hydrostatic equilibrium
  * we showed that this would cause the gas to dominate over dark matter at large radii, violating the constraint $\rho_{\mathrm{gas}}/\rho \to f_{\mathrm{b}}$
    $\Rightarrow$ **isothermal assumption for gas not realistic!**