The Physics of Galaxy Clusters
6th Lecture

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  - we discussed the anisotropic viscous stress tensor and the conductive heat flux

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- **Gas in Dark Matter halo profiles**
  - in the exercises, you played with the NFW profile and derived the corresponding potential
  - we filled gas in the NFW potential and derived the density profile assuming an isothermal gas in hydrostatic equilibrium
  - we showed that this would cause the gas to dominate over dark matter at large radii, violating the constraint $\rho_{\text{gas}}/\rho \rightarrow f_{\text{b}}$
    $\Rightarrow$ **isothermal assumption for gas not realistic!**

Buoyancy Instabilities – 1

- Today, we are studying adiabatic **hydrodynamic perturbations about an atmosphere in hydrostatic equilibrium**. The starting point are conservation equations of mass, momentum, and internal energy (or equivalently entropy) without viscosity and magnetic fields:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = 0, \qquad (1)$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla}) \boldsymbol{v} = -\frac{\boldsymbol{\nabla} P}{\rho} + \boldsymbol{g}, \qquad (2)$$

$$\rho T \frac{\mathrm{d}s}{\mathrm{d}t} = -\boldsymbol{\nabla} \cdot \boldsymbol{Q}. \qquad (3)$$

- Here, $\rho(t, \boldsymbol{x})$ and $\boldsymbol{v}(t, \boldsymbol{x})$ are the density and velocity of the cosmic fluid at position $\boldsymbol{x}$ and time t, $\boldsymbol{g}$ is a conservative force field per unit mass (such as the gravitational acceleration), P is the thermal pressure, T is the temperature, s is the entropy per unit mass and $\mathrm{d}/\mathrm{d}t = \partial/\partial t + \boldsymbol{v} \cdot \boldsymbol{\nabla}$ is a Lagrangian time derivative.

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- We assume that the background plasma is thermally stratified in the presence of a uniform gravitational field in the vertical direction, $\boldsymbol{g} = -g\boldsymbol{e}_z$ where $\boldsymbol{e}_z$ is the unit vector in the z direction.
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- In order for the initial equilibrium to be in hydrostatic steady state, this implies $\mathrm{d}s/\mathrm{d}t = 0$ and hence $\boldsymbol{\nabla} \cdot \boldsymbol{Q}_0 = 0$.
- If we adopt Fick's law for the background heat flux $\boldsymbol{Q}_0 = -\chi\boldsymbol{e}_z \mathrm{d}T_0/\mathrm{d}z$, then

$$\frac{\mathrm{d}^2 T_0}{\mathrm{d}z^2} = 0 \quad \Longrightarrow \quad T_0(z) = a + bz,$$

i.e., the temperature varies at most linearly with height.

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  i.e., the temperature varies at most linearly with height.
- While the steady state assumption is formally required, we note that as long as the time scale for the evolution of the system is longer than the local dynamical time, the general features of the instability described here are unlikely to depend critically on this steady state assumption.

Buoyancy Instabilities – 3

- We perturb the stratified plasma and split the dynamical quantities into background values and small perturbations: $\rho = \rho_0 + \delta\rho$, $\boldsymbol{v} = \delta\boldsymbol{v}$, $P = P_0 + \delta P$, and $s = s_0 + \delta s$. To first order, we obtain for the time derivative of the entropy

$$\frac{\partial s}{\partial t} = \frac{1}{\gamma - 1}\frac{k_\mathrm{B}}{\bar{m}}\frac{\partial(\ln P\rho^{-\gamma})}{\partial t} \tag{4}$$

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- Using Eqn. (4), we obtain to first order for our conservation equations (1) to (3)

$$\frac{\partial\delta\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho_0\delta\boldsymbol{v}) = 0, \tag{5}$$

$$\frac{\partial\delta\boldsymbol{v}}{\partial t} - \frac{\delta\rho}{\rho_0^2}\boldsymbol{\nabla}P_0 + \frac{\boldsymbol{\nabla}\delta P}{\rho_0} = 0, \tag{6}$$

$$\frac{1}{\gamma-1}\left(\frac{\partial\delta P}{\partial t} - \frac{\gamma k_\mathrm{B} T_0}{\bar{m}}\frac{\partial\delta\rho}{\partial t}\right) + \rho_0 T_0(\delta\boldsymbol{v}\cdot\boldsymbol{\nabla})s_0 = -\boldsymbol{\nabla}\cdot\delta\boldsymbol{Q}, \tag{7}$$

where we have used $\boldsymbol{g} = \boldsymbol{\nabla}P_0/\rho_0$ in Eqn. (6).

Buoyancy Instabilities – 4

Transform the first-order conservation equations into Fourier space by decomposing all dynamical variables into plane waves.

- We can decompose all dynamical variables ($\delta\rho$, $\delta\boldsymbol{v}$, δs, δP, $\delta\boldsymbol{Q}$) into plane waves,

$$\delta\rho(\boldsymbol{x}, t) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3}\, \delta\hat{\rho}(\boldsymbol{k}, \omega)\, \mathrm{e}^{-\mathrm{i}\omega t + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}},$$

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- However, as we will see, the growth rate of the perturbations depends in general on position (i.e., height in the gravitational potential), which renders this approach inaccurate after some time because the wave vector will start to depend on position and different wave vectors are not any more linearly independent.
- The wave vector is defined as $\boldsymbol{k} = k_x \boldsymbol{e}_x + k_y \boldsymbol{e}_y + k_z \boldsymbol{e}_z$ and we define $k_\perp^2 = k_x^2 + k_y^2$ to be the wave vector perpendicular to the local gravitational field.
- The **WKB assumption** requires $kH \gg 1$, where H is the local scale height of the system (which is given) and $k = |\boldsymbol{k}|$. Hence you have to choose the wave lengths (and wave numbers) that are on small enough scales (relative to the cluster scale height) for the overall stratification not to matter!

Buoyancy Instabilities – 5

- The first term in the entropy equation is transformed according to:

$$\frac{\partial \delta P}{\partial t} - \frac{\gamma k_\mathrm{B} T_0}{\bar{m}} \frac{\partial \delta \rho}{\partial t} \quad \Longrightarrow \quad -\mathrm{i}\omega \left(\delta \hat{P} - \frac{\gamma k_\mathrm{B} T_0}{\bar{m}} \delta \hat{\rho} \right)$$

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- We can compare both terms by substituting the dispersion relation for sound waves, $\delta \hat{P} / \delta \hat{\rho} = \omega^2 / k^2$ (proof in exercises)

$$\delta \hat{P} = \frac{\omega^2}{k^2} \delta \hat{\rho} \overset{!}{\ll} c_{\mathrm{s}}^2 \delta \hat{\rho} = \frac{\gamma k_{\mathrm{B}} T_0}{\bar{m}} \delta \hat{\rho} \quad \Longrightarrow \quad \omega \overset{!}{\ll} k c_{\mathrm{s}},$$

- The inequalities imply that we discard time scales faster than the sound crossing time (“Boussinesq approximation”), i.e., physics associated with propagating sound waves. Hence, we effectively drop the δP term in the energy equation (but not in the momentum equation).

Buoyancy Instabilities – 6

- Remember, the conservation equations are given by

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$$\frac{\partial \delta \boldsymbol{v}}{\partial t} - \frac{\delta\rho}{\rho_0^2} \boldsymbol{\nabla} P_0 + \frac{\boldsymbol{\nabla} \delta P}{\rho_0} = 0,$$

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- Using the Fourier transformation

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we transform the conservation equations in the Boussinesq approximation,

$$-\mathrm{i}\omega\delta\hat{\rho} + (\delta\hat{\boldsymbol{v}}\cdot\boldsymbol{\nabla})\rho_0 + \mathrm{i}\rho_0 \boldsymbol{k}\cdot\delta\hat{\boldsymbol{v}} = 0, \quad (8)$$
$$-\mathrm{i}\omega\delta\hat{\boldsymbol{v}} - \frac{\delta\hat{\rho}}{\rho_0^2}\boldsymbol{\nabla} P_0 + \mathrm{i}\boldsymbol{k}\frac{\delta\hat{P}}{\rho_0} = 0, \quad (9)$$
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Buoyancy Instabilities – 7

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- The perturbed continuity equation reads in Fourier space

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- An order of magnitude analysis of the perturbed continuity equation yields

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  which follows from the WKB approximation.
- Thus, the last term dominates over the first two terms and leaves us with the near incompressibility condition:

$$\boldsymbol{k}\cdot\delta\hat{\boldsymbol{v}} = 0 \quad \Longrightarrow \quad k_\perp\delta\hat{v}_\perp + k_z\delta\hat{v}_z = 0.$$

- This is a direct consequence of the requirement that perturbations of our stratified hydrostatic background need to remain small so that a Fourier decomposition into plane waves is complete and phases of different k modes grow independently and do not mix.

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Buoyancy Instabilities – 8

- We multiply the perturbed momentum equation by $\boldsymbol{k}$ from the left:

$$\boldsymbol{k}\cdot \left| \quad -\mathrm{i}\omega\delta\hat{\boldsymbol{v}} - \frac{\delta\hat{\rho}}{\rho_0^2}\boldsymbol{\nabla} P_0 + \mathrm{i}\boldsymbol{k}\frac{\delta\hat{P}}{\rho_0} = 0, \right. \tag{11}$$

and obtain

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- Using $\boldsymbol{g} = -g\boldsymbol{e}_z = \boldsymbol{\nabla}P_0/\rho_0$ and $\boldsymbol{e}_z\cdot\boldsymbol{k} = k_z$, we obtain a purely vertical equation:

$$-\mathrm{i}\omega\boldsymbol{k}\cdot\delta\hat{\boldsymbol{v}} = 0 = -\frac{\delta\hat{\rho}}{\rho_0}gk_z - \mathrm{i}k^2\frac{\delta\hat{P}}{\rho_0},$$

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- The perpendicular part of Eqn. (11) is obtained after taking the scalar product with $\boldsymbol{e}_\perp$ and reads ($\boldsymbol{\nabla}P_0$ has only a vertical component)

$$\frac{\delta\hat{P}}{\rho_0} = \omega\frac{\delta\hat{v}_\perp}{k_\perp}.$$

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Buoyancy Instabilities – 9

- Combining the vertical and perpendicular momentum equations

$$-\mathrm{i}\omega\boldsymbol{k}\cdot\delta\hat{\boldsymbol{v}} = 0 \quad = \quad -\frac{\delta\hat{\rho}}{\rho_0} g k_z - \mathrm{i}k^2\frac{\delta\hat{P}}{\rho_0} \qquad \Big| \times \left(\frac{1}{gk_z}\right)$$

$$\text{and} \qquad \frac{\delta\hat{P}}{\rho_0} \quad = \quad \omega\frac{\delta\hat{v}_\perp}{k_\perp}$$

yields

$$\frac{\delta\hat{\rho}}{\rho_0} = -\frac{\mathrm{i}\omega}{g}\frac{k^2}{k_z}\frac{\delta\hat{v}_\perp}{k_\perp} \qquad \text{or} \qquad \mathrm{i}\omega\frac{\delta\hat{\rho}}{\rho_0} = \frac{\omega^2}{g}\frac{k^2}{k_z}\frac{\delta\hat{v}_\perp}{k_\perp}.$$

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- Substituting this into the perturbed entropy equation (assuming $\boldsymbol{k}\cdot\delta\hat{\boldsymbol{Q}} = 0$) gives

$$\begin{aligned}
\mathrm{i}\omega\frac{\gamma}{\gamma-1}P_0\frac{\delta\hat{\rho}}{\rho_0} + \rho_0 T_0(\delta\hat{\boldsymbol{v}}\cdot\boldsymbol{\nabla})s_0 &= 0 \\
\mathrm{i}\omega\frac{\gamma}{\gamma-1}P_0\frac{\delta\hat{\rho}}{\rho_0} + \rho_0 T_0\left(\delta\hat{v}_\perp\cancel{\frac{\partial s_0}{\partial x_\perp}} + \delta\hat{v}_z\frac{\partial s_0}{\partial z}\right) &= 0 \qquad \Big| \times \left(\frac{\gamma-1}{\gamma P_0}\right) \\
\frac{\omega^2}{g}\frac{k^2}{k_z}\frac{\delta\hat{v}_\perp}{k_\perp} + \frac{\gamma-1}{\gamma}\frac{\rho_0 T_0}{P_0}\frac{\partial s_0}{\partial z}\delta\hat{v}_z &= 0 \qquad \Big| \times \left(g\frac{k_z}{k^2}\right) \\
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Buoyancy Instabilities – 10

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$$\omega^2 \frac{\delta \hat{v}_\perp}{k_\perp} + \left(\frac{\gamma - 1}{\gamma} \frac{\bar{m}}{k_\mathrm{B}} g \frac{\partial s_0}{\partial z}\right) \frac{k_z}{k^2} \delta \hat{v}_z = 0.$$

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$$N^2 = \frac{\gamma - 1}{\gamma} \frac{\bar{m}}{k_\mathrm{B}} g \frac{\partial s_0}{\partial z} = \frac{g}{\gamma} \frac{\partial \ln K}{\partial z}, \qquad (12)$$

where $K = P\rho^{-\gamma}$ and use the incompressibility condition:

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- Hence, we obtain the dispersion relation for gravity waves,

$$\begin{aligned} \omega^2 \frac{\delta \hat{v}_\perp}{k_\perp} &= N^2 \frac{k_\perp}{k^2} \delta \hat{v}_\perp, \\ \omega^2 &= N^2 \frac{k_\perp^2}{k^2}. \end{aligned} \qquad (13)$$

AIP

Dispersion relation for gravity waves

The dispersion relation for gravity waves reads:

$$\omega^2 = N^2 \frac{k_\perp^2}{k^2}, \quad N^2 = \frac{g}{\gamma} \frac{\partial \ln K}{\partial z}.$$

and has important consequences:

- For a stably stratified atmosphere where the entropy is increasing outward ($\partial s/\partial z > 0$ or $\partial K/\partial z > 0$), ω is positive and the displaced fluid parcel oscillates with the Brunt-Väisälä frequency, N, around the equilibrium position.

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- Since $k_\perp \leq |\boldsymbol{k}|$, g-modes have a maximum possible frequency of $\omega_{\mathrm{max}} = N$ at which point $k_\perp = |\boldsymbol{k}|$ and $k_z = 0$. If the Brunt-Väisälä frequency is a decreasing function of height z, g-modes of a given frequency ω will be confined/trapped below the height at which $N(z) = \omega$.

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- Note: magnetic fields complicate this picture significantly!

AIP

Vorticity – 1

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- To this end, we introduce the Levi-Civita symbol which is totally antisymmetric in all the indices, i.e., when any two indices are interchanged the symbol is negated

$$\varepsilon_{ijk} = -\varepsilon_{ikj}.$$

Hence, if the Levi-Civita symbol is combined with an expression that is symmetric in any two indices, say $\varepsilon_{ijk} a_j a_k$ then the result must be identical to zero because by exchanging the indices j and k, the expression must be equal to its negative value (which is only possible for zero).

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- Let's prove the property of an incompressible flow. Adopting Einstein's sum convention, i.e., we sum over identical indices, and using the short-hand notation $\partial_i = \partial / \partial x_i$, we obtain:

$$(\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \times \boldsymbol{A})_i = \partial_i \varepsilon_{ijk} \partial_j A_k = \varepsilon_{ijk} \partial_i \partial_j A_k \equiv 0$$

because the expression $\partial_i \partial_j$ is symmetric in i and j.

Vorticity – 2

- Let's prove the identity

$$(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} \equiv \frac{1}{2}\nabla(\boldsymbol{v}^2) - \boldsymbol{v} \times \boldsymbol{\omega},$$

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- This is most easily done within the Levi-Civita formalism and using the Kronecker symbol δ_{ij} which is unity for $i = j$ and zero otherwise. We obtain

$$\begin{aligned}
[\boldsymbol{v} \times (\nabla \times \boldsymbol{v})]_i &= \varepsilon_{ijk} v_j \varepsilon_{klm} \partial_l v_m \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) v_j \partial_l v_m \\
&= v_m \partial_i v_m - v_l \partial_l v_i \\
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- Equivalently, you can show that

$$\begin{aligned}
\nabla\times[(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}] &= -\nabla\times(\boldsymbol{v}\times\boldsymbol{\omega}) \\
&= -(\boldsymbol{\omega}\cdot\nabla)\boldsymbol{v} + \boldsymbol{\omega}(\nabla\cdot\boldsymbol{v}) + (\boldsymbol{v}\cdot\nabla)\boldsymbol{\omega}
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  since $\nabla\cdot\boldsymbol{\omega} \equiv 0$.

Vorticity – 3

Derive the evolution equation for vorticity in general and in the subsonic regime, $\mathcal{M} = v/c_\mathrm{s} \ll 1$ (where $c_\mathrm{s}^2 = \gamma P/\rho$).

AIP

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- If we apply the curl operator to the momentum equation and adopt the definition of vorticity, $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{v}$, we obtain

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{\nabla} \times [(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v}] = -\boldsymbol{\nabla} \times \left(\frac{\boldsymbol{\nabla} P}{\rho} \right) - \boldsymbol{\nabla} \times \boldsymbol{\nabla}\Phi = \frac{1}{\rho^2} \boldsymbol{\nabla}\rho \times \boldsymbol{\nabla} P$$

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- Using the identity on the previous slide, we obtain the vorticity evolution equation,

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- Hence, vorticity production is associated with departures between surfaces of constant density and those of constant pressure. Given that $\boldsymbol{\nabla} P$ is in the vertical direction (as defined by the local gravitational field), the term $\boldsymbol{\nabla} \rho \times \boldsymbol{\nabla} P$ and hence the generated vorticity will lie principally in the horizontal plane.

AIP

Turbulence – 1

- The Navier-Stokes equation in the non-conservative form reads

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = \boldsymbol{g} - \frac{1}{\rho}\boldsymbol{\nabla} P + \frac{1}{\rho}\boldsymbol{\nabla} \cdot \bar{\boldsymbol{\Pi}}.$$

- Assuming a “Newtonian fluid” we have

$$\begin{aligned} \Pi_{ij} &= \eta D_{ij} + \xi \delta_{ij} (\boldsymbol{\nabla} \cdot \boldsymbol{v}), \text{ where} \\ D_{ij} &= \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta_{ij}(\boldsymbol{\nabla} \cdot \boldsymbol{v}) \end{aligned}$$

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- Assuming an incompressible fluid with $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$, we obtain for the x component of the viscous shear force:

$$\begin{aligned} \frac{1}{\eta}(\boldsymbol{\nabla} \cdot \bar{\boldsymbol{\Pi}})_x &= \frac{\partial}{\partial x}\left(2\frac{\partial v_x}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right) \\ &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) v_x = \boldsymbol{\nabla}^2 v_x = \Delta v_x, \end{aligned}$$

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- If we introduce the *kinematic viscosity* $\nu = \eta/\rho$ we obtain

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = \boldsymbol{g} - \frac{1}{\rho}\boldsymbol{\nabla} P + \nu\Delta\boldsymbol{v}.$$

AIP

Turbulence – 2

Explain the physical meaning of the Reynolds number.

- We compare the time scales for advection, $t_{\rm adv}$, and for viscous dissipation, $t_{\rm diss}$:

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- It turns out that the particle mean free path is the typical length over which the fluid can communicate changes in its shear stress. A fluid with a longer mean path length therefore more easily opposes changes to its local shear velocity, i.e., is more viscous.

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  where L and v are characteristic length and velocity scales of the (macroscopic) system.

- We define the Reynolds number to be the ratio of dissipative-to-advective time scale,

  $$\mathrm{Re} = \frac{t_{\rm diss}}{t_{\rm adv}} = \frac{Lv}{\nu} = \frac{L}{\lambda_{\rm mfp}} \frac{v}{v_{\rm th}},$$

  This shows that Re is the product of the ratios of macroscopic-to-microscopic length and velocity scales.

Turbulence – 4

- Note that the assumption of an incompressible flow

$$\boldsymbol{v}(\boldsymbol{x},t) = \int \hat{\boldsymbol{v}}(\boldsymbol{k},\omega)\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}\mathrm{d}^3k\mathrm{d}\omega,$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0 \implies \boldsymbol{k}\cdot\hat{\boldsymbol{v}} = 0$$

does not allow for longitudinal disturbances (sound waves), but only for rotational flows, so-called "eddies" and implies subsonic velocities (since supersonic velocities would cause the formation of shocks, which necessarily have $\boldsymbol{\nabla}\cdot\boldsymbol{v} \neq 0$).

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- In three dimensions, energy is being fed into the turbulent cascade on the macroscopic "injection scale" L with a typical velocity v. Energy is being transported from large to small scales as large eddies break up into smaller eddies, thereby conserving vorticity in the absence of the baroclinic term.

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- The energy transport to small scales continues until the energy is dissipated through the production of viscous heat on the microscopic "viscous" scale, λ_{mfp}, which is of order the particle mean free path. The scales in between, for $\lambda_{\mathrm{mfp}} < \lambda < L$, are called the "inertial subrange".

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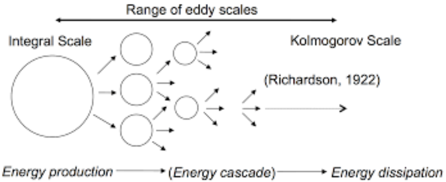
Turulence – 5



- Richardson (1922): "Big whirls have little whirls that feed on their velocity, and little whirls have lesser whirls and so on to viscosity."
- Kolmogorov's theory of turbulence (1941):
  - Turbulence displays universal properties independent of initial and boundary conditions.
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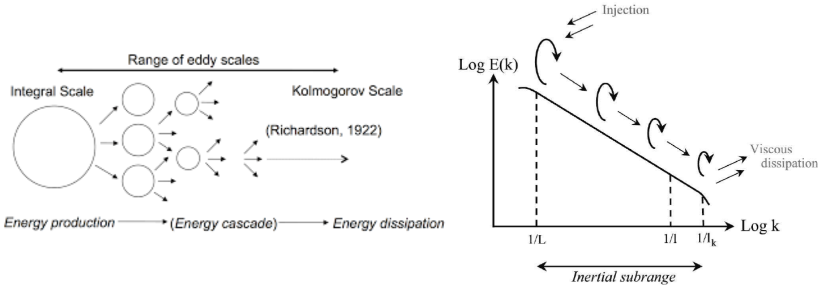

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  - Energy transfer between eddies on intermediate scales is lossless: $\Rightarrow$ kinetic energy spectrum $E(k) \propto k^{-5/3}$ (proof: next slides)

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Turbulence – 6

- Let λ be the size of an eddy and v_λ the typical rotational velocity across the eddy. The energy flow through that scale is the product of kinetic energy and the eddy turnover rate on that scale,

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- The largest eddies assume the highest velocities (and thus the highest kinetic energies), but the smallest eddies have the highest vorticity

$$|\boldsymbol{\omega}| \approx \frac{v_\lambda}{\lambda} \approx \frac{v}{(\lambda^2 L)^{1/3}}.$$

Since the overall vorticity is approximately conserved this implies that turbulence becomes more and more intermittent on smaller scales, i.e., less volume is filled with turbulent eddies.

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Turbulence – 7

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$$P_v \propto \lambda^3 \xi_v \propto k^{-3} \left(\dot{\epsilon} k^{-1}\right)^{2/3} \propto \dot{\epsilon}^{2/3} k^{-11/3}.$$

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- The power per linear and logarithmic interval in k-space scale as

$$\begin{aligned} P_v k^2 \mathrm{d}k &\propto \dot{\epsilon}^{2/3} k^{-5/3} \mathrm{d}k, \quad \text{and} \\ P_v k^3 \mathrm{d}\ln k &\propto \dot{\epsilon}^{2/3} k^{-2/3} \mathrm{d}\ln k, \end{aligned}$$

which is the Kolmogorov turbulence spectrum of driven turbulence.

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Turbulence – 8

We just talked about incompressible, subsonic (Kolmogorov) turbulence. The assumed steady state implies a constant driving mechanism at the outer scale. By contrast, in clusters we encounter decaying turbulence: a merger injects kinetic energy on scales $L \sim r_{\mathrm{c}}$, which will successively decay after a few eddy turnover time scales L/v. The possible implications of turbulence in clusters is mainly

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4. heating cool cores and possible arresting the over-cooling in them, provided the coupling efficiency of $P\mathrm{d}V$ work to the turbulent cascade is high and the dissipation is volume filling and thermally stable.

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- small scales have Re $\sim \frac{L}{\lambda_{\mathrm{mfp}}} \lesssim 2000$: laminar flow
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- turbulent energy cascades from large to small scales
- turbulent diffusion causes horizontal transport and rarefies smoke until it dissolves in the background

- **Buoyancy Instabilities**
  * studied hydrodynamic perturbations in hydrostatic atmosphere
  * atmosphere is stably stratified if entropy increases outwards: perturbations oscillate at Brunt-Väisälä frequency $N \sim g/c_\mathrm{s}$
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- **Turbulence**
  - * flow becomes turbulent if outer and inner (viscous) scales have large separation
  - * incompressible turbulence describes non-linear interactions of rotational motions (eddies)
  - * 3D: energy transport from large to small (viscous) scales at λ_mfp, where kinetic energy is dissipated into heat

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