The Physics of Galaxy Clusters
7th Lecture

Christoph Pfrommer

Leibniz Institute for Astrophysics, Potsdam (AIP)

University of Potsdam

Lectures in the International Astrophysics Masters Program at Potsdam University

- **Buoyancy Instabilities**
  * studied hydrodynamic perturbations in hydrostatic atmosphere
  * atmosphere is stably stratified if entropy increases outwards: perturbations oscillate at Brunt-Väisälä frequency
  * instability if entropy is decreasing outwards: buoyant motions until the entropy profile is inverted and new stable equilibrium is established
  * gravity waves can be trapped if Brunt-Väisälä frequency decreases with height

- **Buoyancy Instabilities**
  * studied hydrodynamic perturbations in hydrostatic atmosphere
  * atmosphere is stably stratified if entropy increases outwards: perturbations oscillate at Brunt-Väisälä frequency
  * instability if entropy is decreasing outwards: buoyant motions until the entropy profile is inverted and new stable equilibrium is established
  * gravity waves can be trapped if Brunt-Väisälä frequency decreases with height
- **Vorticity**
  * vorticity is conserved in subsonic, polytropic ($P = P(\rho)$) flows
  * vorticity produced by baroclinic term ($\boldsymbol{\nabla}\rho \times \boldsymbol{\nabla}P \neq \mathbf{0}$) in horizontal plane
  * at high Mach numbers, compressible motions can also produce vorticity

- **Buoyancy Instabilities**
  * studied hydrodynamic perturbations in hydrostatic atmosphere
  * atmosphere is stably stratified if entropy increases outwards: perturbations oscillate at Brunt-Väisälä frequency
  * instability if entropy is decreasing outwards: buoyant motions until the entropy profile is inverted and new stable equilibrium is established
  * gravity waves can be trapped if Brunt-Väisälä frequency decreases with height
- **Vorticity**
  * vorticity is conserved in subsonic, polytropic ($P = P(\rho)$) flows
  * vorticity produced by baroclinic term ($\boldsymbol{\nabla}\rho \times \boldsymbol{\nabla}P \neq \mathbf{0}$) in horizontal plane
  * at high Mach numbers, compressible motions can also produce vorticity
- **Turbulence**
  * flow becomes turbulent if outer and inner (viscous) scales have large separation
  * incompressible turbulence describes non-linear interactions of rotational motions (eddies)
  * 3D: energy transport from large to small (viscous) scales at λ_{mfp}, where kinetic energy is dissipated into heat

Shock formation – 1

- Imagine the propagation of a sound wave with finite amplitude. The sound speed is higher at higher temperature as $c_\mathrm{s}^2 \propto k_\mathrm{B} T$, so that the crest of the wave gradually overtakes the colder trough ($T \propto \rho^{\gamma-1}$).

AIP

Shock formation – 1

- Imagine the propagation of a sound wave with finite amplitude. The sound speed is higher at higher temperature as $c_s^2 \propto k_B T$, so that the crest of the wave gradually overtakes the colder trough ($T \propto \rho^{\gamma-1}$).
- When faster moving gas overtakes slower gas, we would obtain a multivalued solution that is inconsistent with the hydrodynamic equations. Instead, we get a discontinuous change of density and velocity, a so-called "shock". This steepening happens even for $\gamma = 1$ because of the non-linear dependence on the velocity in the equations.

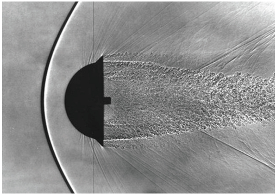
Shock formation – 1

- Imagine the propagation of a sound wave with finite amplitude. The sound speed is higher at higher temperature as $c_s^2 \propto k_B T$, so that the crest of the wave gradually overtakes the colder trough ($T \propto \rho^{\gamma-1}$).
- When faster moving gas overtakes slower gas, we would obtain a multivalued solution that is inconsistent with the hydrodynamic equations. Instead, we get a discontinuous change of density and velocity, a so-called "shock". This steepening happens even for $\gamma = 1$ because of the non-linear dependence on the velocity in the equations.
- Shocks can also be produced by any supersonic compressible disturbance (or through non-linear interactions of subsonic compressible modes). This can result from a supernova explosion within a galaxy, by gas accreting super-sonically onto a cluster, or if two galaxy clusters merge to form a larger entity (as a result of their mutual gravitational interactions).

Shock formation – 2

- In general, a shock wave is
  1. propagating faster than the "signal speed" for compressible waves (i.e., the fastest linear eigenmode of the system which is the sound speed c_s in a hydrodynamic fluid or the fast magnetosonic mode in the high-beta magneto-hydrodynamic plasma of a galaxy cluster),
  2. producing an irreversible change of the fluid state, i.e., an increase in entropy, and
  3. can either be caused by a pressure-driven compressive disturbance, results from non-linear wave interactions, or is caused by supersonic collisions of two streams of fluids.

shock waves: sudden change in density, temperature, and pressure that decelerates supersonic flow.



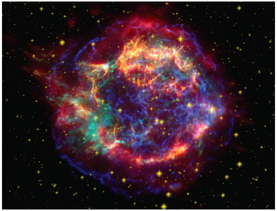
thickness $\sim$ mean free path $\lambda_{\rm mfp}$

in air, $\lambda_{\rm mfp} \sim \mu$m,
on Earth, most shocks are mediated by collisions.

slide concept Spitkovsky

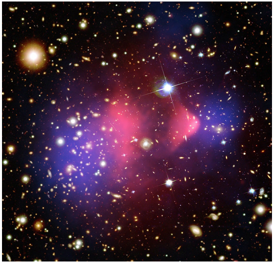
Shock waves

shock waves: sudden change in density, temperature, and pressure that decelerates supersonic flow.



thickness $\sim$ mean free path λ_{mfp}

in air, $\lambda_{\mathrm{mfp}} \sim \mu\mathrm{m}$,
on Earth, most shocks are mediated by collisions.



clusters/galaxies, Coulomb collisions set λ_{mfp}:
$\lambda_{\mathrm{mfp}} \sim L_{\mathrm{cluster}}/10, \qquad \lambda_{\mathrm{mfp}} \sim L_{\mathrm{SNR}}$

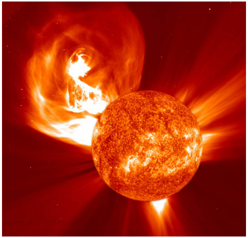
Mean free path $\gg$ observed shock width!

$\rightarrow$ shocks must be mediated without collisions, but through interactions with collective fields
$\rightarrow$ collisionless shocks

slide concept Spitkovsky

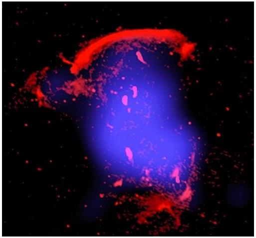
AIP

Astrophysical shocks



solar system shocks $\sim R_\odot$
coronal mass ejection (SOHO)

interstellar shocks $\sim$ 20 pc
supernova 1006 (CXC/Hughes)

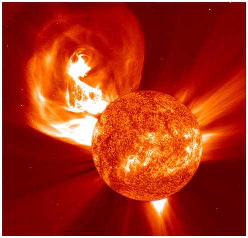


cluster shocks $\sim$ 2 Mpc
giant radio relic (van Weeren)

Astrophysical shocks

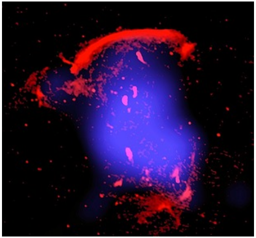
astrophysical collisionless shocks can:

- accelerate particles (electrons and ions) $\rightarrow$ cosmic rays (CRs)
- amplify magnetic fields (or generate them from scratch)
- exchange energy between electrons and ions



solar system shocks $\sim R_\odot$
coronal mass ejection (SOHO)

interstellar shocks $\sim$ 20 pc
supernova 1006 (CXC/Hughes)



cluster shocks $\sim$ 2 Mpc
giant radio relic (van Weeren)

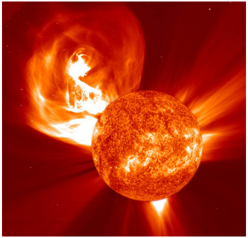
AIP

Astrophysical shocks

astrophysical collisionless shocks can:

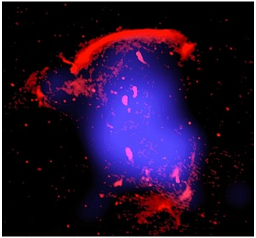
- accelerate particles (electrons and ions) $\rightarrow$ cosmic rays (CRs)
- amplify magnetic fields (or generate them from scratch)
- exchange energy between electrons and ions

collisionless shocks $\Longleftrightarrow$ energetic particles $\Longleftrightarrow$ electro-magnetic waves



solar system shocks $\sim R_\odot$
coronal mass ejection (SOHO)

interstellar shocks $\sim$ 20 pc
supernova 1006 (CXC/Hughes)



cluster shocks $\sim$ 2 Mpc
giant radio relic (van Weeren)

Conservation laws – 1

- To understand fluid discontinuities, we consider the conservation laws of mass, momentum, and internal energy in the absence of external gravitational forces and conductive heat flux (which act on time scale that are much longer in comparison to the transition times at shocks or tangential discontinuities),

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = 0, \quad (1)$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla}) \boldsymbol{v} = -\frac{\boldsymbol{\nabla} P}{\rho} + \frac{1}{\rho} \boldsymbol{\nabla} \cdot \bar{\boldsymbol{\Pi}}, \quad (2)$$

$$\frac{\partial}{\partial t}(\rho \epsilon) + \boldsymbol{\nabla} \cdot (\rho \epsilon \boldsymbol{v}) = -P \boldsymbol{\nabla} \cdot \boldsymbol{v} + \bar{\boldsymbol{\Pi}} : \boldsymbol{\nabla} \boldsymbol{v}. \quad (3)$$

AIP

Conservation laws – 1

- To understand fluid discontinuities, we consider the conservation laws of mass, momentum, and internal energy in the absence of external gravitational forces and conductive heat flux (which act on time scale that are much longer in comparison to the transition times at shocks or tangential discontinuities),

$$\begin{aligned} \frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) &= 0, & (1)\\ \frac{\partial\boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} &= -\frac{\boldsymbol{\nabla}P}{\rho} + \frac{1}{\rho}\boldsymbol{\nabla}\cdot\bar{\boldsymbol{\Pi}}, & (2)\\ \frac{\partial}{\partial t}(\rho\epsilon) + \boldsymbol{\nabla}\cdot(\rho\epsilon\boldsymbol{v}) &= -P\boldsymbol{\nabla}\cdot\boldsymbol{v} + \bar{\boldsymbol{\Pi}}:\boldsymbol{\nabla}\boldsymbol{v}. & (3)\end{aligned}$$

- Since we are interested how the total energy density, $\rho\boldsymbol{v}^2/2+\rho\epsilon$, changes in a given volume, we are supplementing the internal energy equation (Eq. 3) with a conservation law for $\rho\boldsymbol{v}^2/2$. To this end, we consider

$$\frac{\partial}{\partial t}\left(\frac{\rho\boldsymbol{v}^2}{2}\right) = \frac{\boldsymbol{v}^2}{2}\frac{\partial\rho}{\partial t} + \rho\boldsymbol{v}\cdot\frac{\partial\boldsymbol{v}}{\partial t}$$

and substitute Eqs. (1) and (2) to get

$$\frac{\partial}{\partial t}\left(\frac{\rho\boldsymbol{v}^2}{2}\right) = -\frac{\boldsymbol{v}^2}{2}\boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) - \rho\boldsymbol{v}\cdot(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} - \boldsymbol{v}\cdot\boldsymbol{\nabla}P + \boldsymbol{v}\cdot(\boldsymbol{\nabla}\cdot\bar{\boldsymbol{\Pi}}).$$

Conservation laws – 2

- Let's recap the last equation,

$$\frac{\partial}{\partial t}\left(\frac{\rho \boldsymbol{v}^2}{2}\right) = -\frac{\boldsymbol{v}^2}{2}\boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) - \rho\boldsymbol{v}\cdot(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} - \boldsymbol{v}\cdot\boldsymbol{\nabla}P + \boldsymbol{v}\cdot(\boldsymbol{\nabla}\cdot\bar{\boldsymbol{\Pi}}).$$

- Using the identity $(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} \equiv \boldsymbol{\nabla}\boldsymbol{v}^2/2$, we obtain an equation for the conservation of kinetic energy density

$$\frac{\partial}{\partial t}\left(\frac{\rho \boldsymbol{v}^2}{2}\right) + \boldsymbol{\nabla}\cdot\left(\frac{1}{2}\rho\boldsymbol{v}^2\boldsymbol{v}\right) = -\boldsymbol{v}\cdot\boldsymbol{\nabla}P + \boldsymbol{v}\cdot(\boldsymbol{\nabla}\cdot\bar{\boldsymbol{\Pi}}). \qquad (4)$$

AIP

- Let's recap the last equation,

$$\frac{\partial}{\partial t}\left(\frac{\rho \boldsymbol{v}^2}{2}\right) = -\frac{\boldsymbol{v}^2}{2}\boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) - \rho\boldsymbol{v}\cdot(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} - \boldsymbol{v}\cdot\boldsymbol{\nabla}P + \boldsymbol{v}\cdot(\boldsymbol{\nabla}\cdot\bar{\boldsymbol{\Pi}}).$$

- Using the identity $(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} \equiv \boldsymbol{\nabla}\boldsymbol{v}^2/2$, we obtain an equation for the conservation of kinetic energy density

$$\frac{\partial}{\partial t}\left(\frac{\rho \boldsymbol{v}^2}{2}\right) + \boldsymbol{\nabla}\cdot\left(\frac{1}{2}\rho\boldsymbol{v}^2\boldsymbol{v}\right) = -\boldsymbol{v}\cdot\boldsymbol{\nabla}P + \boldsymbol{v}\cdot(\boldsymbol{\nabla}\cdot\bar{\boldsymbol{\Pi}}). \qquad (4)$$

- Using another identity $\boldsymbol{\nabla}\cdot(\boldsymbol{v}P) \equiv \boldsymbol{v}\cdot\boldsymbol{\nabla}P + P\boldsymbol{\nabla}\cdot\boldsymbol{v}$, we can rewrite the equation for the conservation of kinetic energy density

$$\frac{\partial}{\partial t}\left(\frac{\rho \boldsymbol{v}^2}{2}\right) + \boldsymbol{\nabla}\cdot\left(\frac{1}{2}\rho\boldsymbol{v}^2\boldsymbol{v} + \boldsymbol{v}P\right) = P\boldsymbol{\nabla}\cdot\boldsymbol{v} + \boldsymbol{v}\cdot(\boldsymbol{\nabla}\cdot\bar{\boldsymbol{\Pi}}),$$

i.e., the kinetic energy is conserved except for adiabatic losses due to converging flows and work done by viscous shear forces.

AIP

Conservation laws – 3

- Let's recap the last equation for the kinetic energy,

$$\frac{\partial}{\partial t}\left(\frac{\rho \boldsymbol{v}^2}{2}\right) + \boldsymbol{\nabla}\cdot\left(\frac{1}{2}\rho \boldsymbol{v}^2 \boldsymbol{v} + \boldsymbol{v}P\right) = P\boldsymbol{\nabla}\cdot\boldsymbol{v} + \boldsymbol{v}\cdot(\boldsymbol{\nabla}\cdot\bar{\boldsymbol{\Pi}}).$$

- Combining this equation with the internal energy equation (Eq. 3),

$$\frac{\partial}{\partial t}(\rho\epsilon) + \boldsymbol{\nabla}\cdot(\rho\epsilon\boldsymbol{v}) \quad = \quad -P\boldsymbol{\nabla}\cdot\boldsymbol{v} + \bar{\boldsymbol{\Pi}} : \boldsymbol{\nabla}\boldsymbol{v}$$

  we derive the equation for total energy conservation

$$\frac{\partial}{\partial t}\left(\rho\epsilon + \frac{\rho \boldsymbol{v}^2}{2}\right) + \boldsymbol{\nabla}\cdot\left\{\boldsymbol{v}\cdot\left[\left(\rho\epsilon + P + \frac{1}{2}\rho\boldsymbol{v}^2\right)\bar{\boldsymbol{1}} + \bar{\boldsymbol{\Pi}}\right]\right\} = 0. \qquad (5)$$

Shocks – 1

- Consider a propagating fluid discontinuity in the rest frame of the discontinuity. Fluid moves from upstream to downstream. We denote the *upstream conditions* by ρ_1, v_1, T_1 and *downstream conditions* by ρ_2, v_2, T_2.

Shocks – 1

- Consider a propagating fluid discontinuity in the rest frame of the discontinuity. Fluid moves from upstream to downstream. We denote the *upstream conditions* by ρ_1, v_1, T_1 and *downstream conditions* by ρ_2, v_2, T_2.
- We would like to derive the relations (also known as “jump conditions”) between ρ_1, v_1, T_1 and ρ_2, v_2, T_2 for a steady-state, plane-parallel geometry of a fluid discontinuity such as a shock. First, we assume that the velocity is perpendicular to the surface of the discontinuity. While this may seem to be a substantial loss of generality, it captures the main effect of discontinuities as we will see by generalizing this simplification in the last part of this section.

Shocks – 1

- Consider a propagating fluid discontinuity in the rest frame of the discontinuity. Fluid moves from upstream to downstream. We denote the *upstream conditions* by ρ_1, v_1, T_1 and *downstream conditions* by ρ_2, v_2, T_2.
- We would like to derive the relations (also known as "jump conditions") between ρ_1, v_1, T_1 and ρ_2, v_2, T_2 for a steady-state, plane-parallel geometry of a fluid discontinuity such as a shock. First, we assume that the velocity is perpendicular to the surface of the discontinuity. While this may seem to be a substantial loss of generality, it captures the main effect of discontinuities as we will see by generalizing this simplification in the last part of this section.
- There are two types of discontinuities:
    1. shocks that are characterized by a mass flux through their interface, and
    2. tangential discontinuities which have *no* mass flux through their interface.

Shocks – 2

If a shock is a true discontinuity in hydrodynamic quantities, are the partial derivatives in our evolution equations for mass, momentum and energy well defined at the shock? How does one deal with this issue in practice? What is really happening at a shock?

Shocks – 2

If a shock is a true discontinuity in hydrodynamic quantities, are the partial derivatives in our evolution equations for mass, momentum and energy well defined at the shock? How does one deal with this issue in practice? What is really happening at a shock?

- Within the shock front or "transition layer" on the scale of the effective mean free path $\lambda_{\rm eff}$, viscous effects are important and cause the shock in the first place, i.e., dissipate kinetic energy and thus generate heat and entropy:

  collisional shocks: $\lambda_{\rm eff} \sim \lambda_{\rm mfp}$

  collisionless shocks: $\lambda_{\rm eff} \ll \lambda_{\rm mfp}$

AIP

Shocks – 2

If a shock is a true discontinuity in hydrodynamic quantities, are the partial derivatives in our evolution equations for mass, momentum and energy well defined at the shock? How does one deal with this issue in practice? What is really happening at a shock?

- Within the shock front or "transition layer" on the scale of the effective mean free path λ_{eff}, viscous effects are important and cause the shock in the first place, i.e., dissipate kinetic energy and thus generate heat and entropy:

  collisional shocks: $\lambda_{\text{eff}} \sim \lambda_{\text{mfp}}$

  collisionless shocks: $\lambda_{\text{eff}} \ll \lambda_{\text{mfp}}$

- In principle, by resolving the viscous/collisionless shock transition layer at scales $\lambda \lesssim \lambda_{\text{mfp}}$ our partial derivatives would be well defined. But on scales $\lambda < \lambda_{\text{mfp}}$ the fluid description breaks down and we have to reside to kinetic theory! How do we proceed here?

Shocks – 2

If a shock is a true discontinuity in hydrodynamic quantities, are the partial derivatives in our evolution equations for mass, momentum and energy well defined at the shock? How does one deal with this issue in practice? What is really happening at a shock?

- Within the shock front or "transition layer" on the scale of the effective mean free path λ_{eff}, viscous effects are important and cause the shock in the first place, i.e., dissipate kinetic energy and thus generate heat and entropy:

  collisional shocks: $\lambda_{\text{eff}} \sim \lambda_{\text{mfp}}$

  collisionless shocks: $\lambda_{\text{eff}} \ll \lambda_{\text{mfp}}$

- In principle, by resolving the viscous/collisionless shock transition layer at scales $\lambda \lesssim \lambda_{\text{mfp}}$ our partial derivatives would be well defined. But on scales $\lambda < \lambda_{\text{mfp}}$ the fluid description breaks down and we have to reside to kinetic theory! How do we proceed here?
- Hence, we must resort to the integrated equations and exchange the fluxes Q (of mass, momentum and energy) across our discretized fluid elements.
- Outside the layer, viscous effects are small on scales $\lambda \gg \lambda_{\text{eff}}$. We will derive conservation equations of the form

$$\frac{\mathrm{d}}{\mathrm{d}x} Q(\rho, v, P) = 0 \implies Q(\rho, v, P) = \text{const.}$$

  and although Q involves viscous terms, we can ignore these outside the shock zone and can derive jump conditions from equations without viscosity terms.

Shocks – 3

- Let's recap the four conservation equations (for mass, momentum, internal and kinetic energy)

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho \boldsymbol{v}) &= 0, \\
\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} &= -\frac{\boldsymbol{\nabla} P}{\rho} + \frac{1}{\rho}\boldsymbol{\nabla}\cdot\bar{\boldsymbol{\Pi}}, \\
\frac{\partial}{\partial t}(\rho\epsilon) + \boldsymbol{\nabla}\cdot(\rho\epsilon\boldsymbol{v}) &= -P\boldsymbol{\nabla}\cdot\boldsymbol{v} + \bar{\boldsymbol{\Pi}} : \boldsymbol{\nabla}\boldsymbol{v}, \\
\frac{\partial}{\partial t}\left(\frac{\rho \boldsymbol{v}^2}{2}\right) + \boldsymbol{\nabla}\cdot\left(\frac{1}{2}\rho\boldsymbol{v}^2\boldsymbol{v}\right) &= -\boldsymbol{v}\cdot\boldsymbol{\nabla}P + \boldsymbol{v}\cdot(\boldsymbol{\nabla}\cdot\bar{\boldsymbol{\Pi}}).
\end{aligned}
$$

AIP

Shocks – 3

- Let's recap the four conservation equations (for mass, momentum, internal and kinetic energy)

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho \boldsymbol{v}) &= 0, \\
\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} &= -\frac{\boldsymbol{\nabla} P}{\rho} + \frac{1}{\rho}\boldsymbol{\nabla}\cdot\bar{\boldsymbol{\Pi}}, \\
\frac{\partial}{\partial t}(\rho\epsilon) + \boldsymbol{\nabla}\cdot(\rho\epsilon\boldsymbol{v}) &= -P\boldsymbol{\nabla}\cdot\boldsymbol{v} + \bar{\boldsymbol{\Pi}} : \boldsymbol{\nabla}\boldsymbol{v}, \\
\frac{\partial}{\partial t}\left(\frac{\rho \boldsymbol{v}^2}{2}\right) + \boldsymbol{\nabla}\cdot\left(\frac{1}{2}\rho\boldsymbol{v}^2\boldsymbol{v}\right) &= -\boldsymbol{v}\cdot\boldsymbol{\nabla}P + \boldsymbol{v}\cdot(\boldsymbol{\nabla}\cdot\bar{\boldsymbol{\Pi}}).
\end{aligned}
$$

- We assume steady state ($\partial/\partial t = 0$) and plane-parallel geometry ($\partial/\partial y = \partial/\partial z = 0$, $\partial/\partial x = \mathrm{d}/\mathrm{d}x$). The conservation laws simplify to

$$
\frac{\mathrm{d}}{\mathrm{d}x}(\rho v) = 0, \tag{6}
$$

$$
v\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}x} + \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}x}\left\{\left(\frac{4}{3}\eta + \xi\right)\frac{\mathrm{d}v}{\mathrm{d}x}\right\}, \tag{7}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}x}(\rho\epsilon v) = -P\frac{\mathrm{d}v}{\mathrm{d}x} + \left(\frac{4}{3}\eta + \xi\right)\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)^2, \tag{8}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}\rho v^2 v\right) = -v\frac{\mathrm{d}P}{\mathrm{d}x} + v\frac{\mathrm{d}}{\mathrm{d}x}\left\{\left(\frac{4}{3}\eta + \xi\right)\frac{\mathrm{d}v}{\mathrm{d}x}\right\}. \tag{9}
$$

Shocks – 4

- The equation for mass conservation (Eq. 6) gives

$$\rho v = \text{const.} \implies \rho_1 v_1 = \rho_2 v_2 = j \implies [\rho v] = 0, \tag{10}$$

where j is the current density and the brackets, $[\ldots]$, indicate differences between the up- and downstream quantities. Note, that the up- and downstream velocities, v_1 and v_2, are measured in the frame of the discontinuity!

AIP

Shocks – 4

- The equation for mass conservation (Eq. 6) gives

$$\rho v = \text{const.} \implies \rho_1 v_1 = \rho_2 v_2 = j \implies [\rho v] = 0, \tag{10}$$

where j is the current density and the brackets, $[\ldots]$, indicate differences between the up- and downstream quantities. Note, that the up- and downstream velocities, v_1 and v_2, are measured in the frame of the discontinuity!

- Using

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\rho v^2\right) = \rho v \frac{\mathrm{d}v}{\mathrm{d}x} + v \frac{\mathrm{d}}{\mathrm{d}x}(\rho v) \overset{(10)}{=} \rho v \frac{\mathrm{d}v}{\mathrm{d}x}$$

allows Eq. (7) to be rewritten as

$$\begin{aligned} &\rho v \frac{\mathrm{d}v}{\mathrm{d}x} + \frac{\mathrm{d}P}{\mathrm{d}x} - \frac{\mathrm{d}}{\mathrm{d}x}\left\{\left(\frac{4}{3}\eta + \xi\right)\frac{\mathrm{d}v}{\mathrm{d}x}\right\} \\ &\quad= \frac{\mathrm{d}}{\mathrm{d}x}\left\{\rho v^2 + P - \left(\frac{4}{3}\eta + \xi\right)\frac{\mathrm{d}v}{\mathrm{d}x}\right\} = 0 \\ &\quad\implies \left[\rho v^2 + P - \left(\frac{4}{3}\eta + \xi\right)\frac{\mathrm{d}v}{\mathrm{d}x}\right] = 0 \end{aligned}$$

Shocks – 4

- The equation for mass conservation (Eq. 6) gives

$$\rho v = \text{const.} \implies \rho_1 v_1 = \rho_2 v_2 = j \implies [\rho v] = 0, \tag{10}$$

where j is the current density and the brackets, $[\dots]$, indicate differences between the up- and downstream quantities. Note, that the up- and downstream velocities, v_1 and v_2, are measured in the frame of the discontinuity!

- Using

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\rho v^2\right) = \rho v \frac{\mathrm{d}v}{\mathrm{d}x} + v \frac{\mathrm{d}}{\mathrm{d}x}(\rho v) \overset{(10)}{=} \rho v \frac{\mathrm{d}v}{\mathrm{d}x}$$

allows Eq. (7) to be rewritten as

$$\begin{aligned} \rho v \frac{\mathrm{d}v}{\mathrm{d}x} + \frac{\mathrm{d}P}{\mathrm{d}x} - \frac{\mathrm{d}}{\mathrm{d}x}\left\{\left(\frac{4}{3}\eta + \xi\right)\frac{\mathrm{d}v}{\mathrm{d}x}\right\} \\ = \frac{\mathrm{d}}{\mathrm{d}x}\left\{\rho v^2 + P - \left(\frac{4}{3}\eta + \xi\right)\frac{\mathrm{d}v}{\mathrm{d}x}\right\} = 0 \\ \implies \left[\rho v^2 + P - \left(\frac{4}{3}\eta + \xi\right)\frac{\mathrm{d}v}{\mathrm{d}x}\right] = 0 \end{aligned}$$

- This demonstrates that within the transition zone (where η, ξ, and $\mathrm{d}v/\mathrm{d}x$ are non-zero) $\rho v^2 + P \neq \text{const.}$ However, in the pre- and post-shock zones, η, ξ, and $\mathrm{d}v/\mathrm{d}x$ are negligible, implying

$$\left[\rho v^2 + P\right] = 0.$$

Shocks – 5

- In the shock transition zone, viscous forces dissipate the incoming kinetic energy into thermal energy and generate entropy.
- Adding the equations for thermal and kinetic energy, Eqs. (8) and (9), yields (for the region outside the transition zone)

$$0 = \frac{\mathrm{d}}{\mathrm{d}x}\left\{v\left(\frac{1}{2}\rho v^2 + \rho\epsilon\right) + Pv\right\} = \frac{\mathrm{d}}{\mathrm{d}x}\left\{\rho v\left(\frac{1}{2}v^2 + \epsilon + \frac{P}{\rho}\right)\right\}$$
$$= \left(\frac{1}{2}v^2 + \epsilon + \frac{P}{\rho}\right)\frac{\mathrm{d}}{\mathrm{d}x}(\rho v) + \rho v \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2 + \epsilon + \frac{P}{\rho}\right).$$

Shocks – 5

- In the shock transition zone, viscous forces dissipate the incoming kinetic energy into thermal energy and generate entropy.
- Adding the equations for thermal and kinetic energy, Eqs. (8) and (9), yields (for the region outside the transition zone)

$$0 = \frac{\mathrm{d}}{\mathrm{d}x}\left\{v\left(\frac{1}{2}\rho v^2 + \rho\epsilon\right) + Pv\right\} = \frac{\mathrm{d}}{\mathrm{d}x}\left\{\rho v\left(\frac{1}{2}v^2 + \epsilon + \frac{P}{\rho}\right)\right\}$$
$$= \left(\frac{1}{2}v^2 + \epsilon + \frac{P}{\rho}\right)\frac{\mathrm{d}}{\mathrm{d}x}(\rho v) + \rho v \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2 + \epsilon + \frac{P}{\rho}\right).$$

Since $\mathrm{d}(\rho v)/\mathrm{d}x = 0$ and $\rho v \neq 0$ for a shock, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2 + \epsilon + \frac{P}{\rho}\right) = 0 \quad \Longrightarrow \quad \left[\frac{1}{2}v^2 + \epsilon + \frac{P}{\rho}\right] = 0.$$

Shocks – 6

- Summarizing, we have the Rankine-Hugoniot jump conditions for a plane-parallel shock in the shock rest frame:

$$[\rho v] = 0, \quad (11)$$

$$\left[\rho v^2 + P\right] = 0, \quad (12)$$

$$\left[\frac{1}{2}v^2 + \epsilon + \frac{P}{\rho}\right] = 0. \quad (13)$$

AIP

Shocks – 6

- Summarizing, we have the Rankine-Hugoniot jump conditions for a plane-parallel shock in the shock rest frame:

$$[\rho v] = 0, \quad (11)$$

$$\left[\rho v^2 + P\right] = 0, \quad (12)$$

$$\left[\frac{1}{2}v^2 + \epsilon + \frac{P}{\rho}\right] = 0. \quad (13)$$

- Independent of the complicated physics within the transition layer, these conditions simply follow from the conservation laws. The first follows from mass conservation, the second from mass and momentum conservation, and the third from mass and total energy conservation.

Shocks – 6

- Summarizing, we have the Rankine-Hugoniot jump conditions for a plane-parallel shock in the shock rest frame:

$$[\rho v] = 0, \quad (11)$$

$$\left[\rho v^2 + P\right] = 0, \quad (12)$$

$$\left[\frac{1}{2} v^2 + \epsilon + \frac{P}{\rho}\right] = 0. \quad (13)$$

- Independent of the complicated physics within the transition layer, these conditions simply follow from the conservation laws. The first follows from mass conservation, the second from mass and momentum conservation, and the third from mass and total energy conservation.
- Using $\epsilon_i = P_i/\{\rho_i(\gamma_i - 1)\}$, we can rewrite the energy jump condition to get

$$\frac{1}{2} v_1^2 + \frac{\gamma_1}{\gamma_1 - 1}\frac{P_1}{\rho_1} = \frac{1}{2} v_2^2 + \frac{\gamma_2}{\gamma_2 - 1}\frac{P_2}{\rho_2}$$

for a single-species gas that is described by a polytropic equation of state. In principle, $\gamma_1 \neq \gamma_2$, since a shock can e.g., dissociate molecules, or raise T so that previously inaccessible degrees of freedom become accessible.

Tangential discontinuities

How does a tangential discontinuity differ from a shock? Name at least two of the three differences.

Tangential discontinuities

How does a tangential discontinuity differ from a shock? Name at least two of the three differences.

- $[\rho v] = 0$ allows for two types of solutions. The first type is clearly $\rho_1 v_1 = \rho_2 v_2 = 0$ and since ρ_1 and ρ_2 are non-zero, we have

$$v_1 = v_2 = 0,$$

$$[\rho v^2 + P] = 0 \quad \Longrightarrow \quad P_1 = P_2 \quad \Longrightarrow \quad [P] = 0$$

which follows from Eq. (12).

Tangential discontinuities

How does a tangential discontinuity differ from a shock? Name at least two of the three differences.

- $[\rho v] = 0$ allows for two types of solutions. The first type is clearly $\rho_1 v_1 = \rho_2 v_2 = 0$ and since ρ_1 and ρ_2 are non-zero, we have

$$v_1 = v_2 = 0,$$
$$[\rho v^2 + P] = 0 \quad \Longrightarrow \quad P_1 = P_2 \quad \Longrightarrow \quad [P] = 0$$

  which follows from Eq. (12).
- The constancy of the normal component of the velocity across such an interface implies that there is no mass flux $j = \rho v$ through a tangential discontinuity. If additionally the tangential velocity is also continuous, a special discontinuity is present which is called a contact discontinuity.

Tangential discontinuities

How does a tangential discontinuity differ from a shock? Name at least two of the three differences.

- $[\rho v] = 0$ allows for two types of solutions. The first type is clearly $\rho_1 v_1 = \rho_2 v_2 = 0$ and since ρ_1 and ρ_2 are non-zero, we have

$$\begin{aligned} & v_1 = v_2 = 0, \\ [\rho v^2 + P] = 0 \quad \Longrightarrow \quad & P_1 = P_2 \quad \Longrightarrow \quad [P] = 0 \end{aligned}$$

  which follows from Eq. (12).
- The constancy of the normal component of the velocity across such an interface implies that there is no mass flux $j = \rho v$ through a tangential discontinuity. If additionally the tangential velocity is also continuous, a special discontinuity is present which is called a contact discontinuity.
- At a tangential discontinuity, there can be an arbitrary jump of density, that however needs to be compensated by the same jump of T, but in the opposite direction (because $P = n\, k_\mathrm{B}\, T$ is constant across a tangential discontinuity)!

Shock Mach number – 1

- The other type of solution requires $\rho_1 v_1 \neq 0$ so that we have a mass flux through this type of discontinuity that we call a "shock".

Shock Mach number – 1

- The other type of solution requires $\rho_1 v_1 \neq 0$ so that we have a mass flux through this type of discontinuity that we call a "shock".
- We define a dimensionless number that characterizes the shock strength, the Mach number as the ratio of shock speed to upstream sound speed $c_1^2 = \gamma P_1/\rho_1$,

$$\mathcal{M}_1 \equiv \frac{v_1}{c_1} = \sqrt{\frac{\rho_1 v_1^2}{\gamma P_1}} = \sqrt{\frac{\bar{m} v_1^2}{\gamma k_{\mathrm{B}} T_1}}, \tag{14}$$

which can be interpreted as a ratio of ram pressure ($\rho_1 v_1^2$)-to-thermal pressure in the pre-shock gas or equivalently a ratio of kinetic-to-thermal energy density.

AIP

Shock Mach number – 1

- The other type of solution requires $\rho_1 v_1 \neq 0$ so that we have a mass flux through this type of discontinuity that we call a "shock".
- We define a dimensionless number that characterizes the shock strength, the Mach number as the ratio of shock speed to upstream sound speed $c_1^2 = \gamma P_1/\rho_1$,

$$\mathcal{M}_1 \equiv \frac{v_1}{c_1} = \sqrt{\frac{\rho_1 v_1^2}{\gamma P_1}} = \sqrt{\frac{\bar{m} v_1^2}{\gamma k_\mathrm{B} T_1}}, \tag{14}$$

which can be interpreted as a ratio of ram pressure ($\rho_1 v_1^2$)-to-thermal pressure in the pre-shock gas or equivalently a ratio of kinetic-to-thermal energy density.
- We can rewrite the Rankine-Hugoniot jump conditions in terms of $\mathcal{M}_1$ (and assuming $\gamma_1 = \gamma_2 = \gamma$ which is applicable for the ionized plasma of the ICM)

$$\frac{\rho_2}{\rho_1} = \frac{v_1}{v_2} = \frac{(\gamma+1)\mathcal{M}_1^2}{(\gamma-1)\mathcal{M}_1^2+2} \xrightarrow{\gamma=1} \mathcal{M}_1^2 \tag{15}$$

$$\frac{P_2}{P_1} = \frac{\rho_2 k_\mathrm{B} T_2}{\rho_1 k_\mathrm{B} T_1} = \frac{2\gamma\mathcal{M}_1^2-(\gamma-1)}{\gamma+1} \xrightarrow{\gamma=1} \mathcal{M}_1^2 \tag{16}$$

$$\frac{T_2}{T_1} = \frac{\left[(\gamma-1)\mathcal{M}_1^2+2\right]\left[2\gamma\mathcal{M}_1^2-(\gamma-1)\right]}{(\gamma+1)^2\mathcal{M}_1^2} \xrightarrow{\gamma=1} 1 \tag{17}$$

Note, the brackets in these equations retrieve their usual meaning.

Shock Mach number – 2

- Let's recap the Rankine-Hugoniot jump conditions:

$$\frac{\rho_2}{\rho_1} = \frac{v_1}{v_2} = \frac{(\gamma+1)\mathcal{M}_1^2}{(\gamma-1)\mathcal{M}_1^2+2}$$

$$\frac{P_2}{P_1} = \frac{\rho_2 k_\mathrm{B} T_2}{\rho_1 k_\mathrm{B} T_1} = \frac{2\gamma\mathcal{M}_1^2-(\gamma-1)}{\gamma+1}$$

$$\frac{T_2}{T_1} = \frac{\left[(\gamma-1)\mathcal{M}_1^2+2\right]\left[2\gamma\mathcal{M}_1^2-(\gamma-1)\right]}{(\gamma+1)^2\mathcal{M}_1^2}$$

Shock Mach number – 2

- Let's recap the Rankine-Hugoniot jump conditions:

$$\frac{\rho_2}{\rho_1} = \frac{v_1}{v_2} = \frac{(\gamma+1)\mathcal{M}_1^2}{(\gamma-1)\mathcal{M}_1^2+2}$$

$$\frac{P_2}{P_1} = \frac{\rho_2 k_\mathrm{B} T_2}{\rho_1 k_\mathrm{B} T_1} = \frac{2\gamma\mathcal{M}_1^2-(\gamma-1)}{\gamma+1}$$

$$\frac{T_2}{T_1} = \frac{\left[(\gamma-1)\mathcal{M}_1^2+2\right]\left[2\gamma\mathcal{M}_1^2-(\gamma-1)\right]}{(\gamma+1)^2\mathcal{M}_1^2}$$

- Those relations simplify for strong shocks ($\mathcal{M}_1 \gg 1$), yielding

$$\frac{\rho_2}{\rho_1} = \frac{v_1}{v_2} \approx \frac{\gamma+1}{\gamma-1} = 4,$$

$$P_2 \approx \frac{2\gamma}{\gamma+1}\mathcal{M}_1^2 P_1 = \frac{2}{\gamma+1}\rho_1 v_1^2 = \frac{3}{4}\rho_1 v_1^2,$$

$$k_\mathrm{B} T_2 \approx \frac{2\gamma(\gamma-1)}{(\gamma+1)^2} k_\mathrm{B} T_1 \mathcal{M}_1^2 = \frac{2(\gamma-1)}{(\gamma+1)^2}\bar{m}v_1^2 = \frac{3}{16}\bar{m}v_1^2,$$

where we used a non-relativistic ideal gas ($\gamma = 5/3$) in the last equalities.

Shock Mach number – 3

- In the shock rest frame, the post-shock kinetic and thermal specific energies are ($\gamma = 5/3$, $\mathcal{M} \gg 1$)

$$v_2 = \frac{v_1}{4} \quad \Longrightarrow \quad \frac{1}{2} v_2^2 \approx \frac{1}{32} v_1^2,$$

$$k_\mathrm{B} T_2 = \frac{3}{16} \bar{m} v_1^2 \quad \Longrightarrow \quad \frac{3}{2} \frac{k_\mathrm{B} T_2}{\bar{m}} \approx \frac{9}{32} v_1^2 = \frac{9}{16} \left(\frac{1}{2} v_1^2 \right).$$

AIP

Shock Mach number – 3

- In the shock rest frame, the post-shock kinetic and thermal specific energies are ($\gamma = 5/3$, $\mathcal{M} \gg 1$)

$$v_2 = \frac{v_1}{4} \quad \Longrightarrow \quad \frac{1}{2}v_2^2 \approx \frac{1}{32}v_1^2,$$

$$k_\mathrm{B}T_2 = \frac{3}{16}\bar{m}v_1^2 \quad \Longrightarrow \quad \frac{3}{2}\frac{k_\mathrm{B}T_2}{\bar{m}} \approx \frac{9}{32}v_1^2 = \frac{9}{16}\left(\frac{1}{2}v_1^2\right).$$

- So roughly half of the pre-shock kinetic energy is converted to thermal energy (in the shock rest frame). The total specific energy ϵ_tot of the post-shock gas is

$$\epsilon_\mathrm{tot,2} = \frac{1}{2}v_2^2 + \frac{3}{2}\frac{k_\mathrm{B}T_2}{\bar{m}} \approx \frac{10}{16}\left(\frac{1}{2}v_1^2\right) = \frac{5}{8}\epsilon_\mathrm{kin,1} = \frac{5}{8}\epsilon_\mathrm{tot,1}$$

  because in a strong shock, the upstream thermal energy is negligible in comparison to the kinetic energy. Hence, $\epsilon_\mathrm{tot,2}$ is lower than $\epsilon_\mathrm{tot,1}$ (in the shock rest frame): what about energy conservation?

Shock Mach number – 3

- In the shock rest frame, the post-shock kinetic and thermal specific energies are ($\gamma = 5/3$, $\mathcal{M} \gg 1$)

$$v_2 = \frac{v_1}{4} \quad \Longrightarrow \quad \frac{1}{2} v_2^2 \approx \frac{1}{32} v_1^2,$$

$$k_\mathrm{B} T_2 = \frac{3}{16} \bar{m} v_1^2 \quad \Longrightarrow \quad \frac{3}{2} \frac{k_\mathrm{B} T_2}{\bar{m}} \approx \frac{9}{32} v_1^2 = \frac{9}{16} \left(\frac{1}{2} v_1^2 \right).$$

- So roughly half of the pre-shock kinetic energy is converted to thermal energy (in the shock rest frame). The total specific energy ϵ_tot of the post-shock gas is

$$\epsilon_{\mathrm{tot},2} = \frac{1}{2} v_2^2 + \frac{3}{2} \frac{k_\mathrm{B} T_2}{\bar{m}} \approx \frac{10}{16} \left(\frac{1}{2} v_1^2 \right) = \frac{5}{8} \epsilon_{\mathrm{kin},1} = \frac{5}{8} \epsilon_{\mathrm{tot},1}$$

  because in a strong shock, the upstream thermal energy is negligible in comparison to the kinetic energy. Hence, $\epsilon_{\mathrm{tot},2}$ is lower than $\epsilon_{\mathrm{tot},1}$ (in the shock rest frame): what about energy conservation?

- The difference is explained by the $P\mathrm{d}V$ work done by pressure and viscosity on the post-shock gas in compressing its volume. Note that this $P\mathrm{d}V$ term is absent in the rest frame of the post-shock gas.

Shock Mach number – 4

- The post-shock Mach number is

$$\mathcal{M}_2 \equiv \frac{v_2}{c_2} = \frac{v_1}{c_1}\frac{v_2}{v_1}\frac{c_1}{c_2} = \mathcal{M}_1 \frac{v_2}{v_1}\left(\frac{T_1}{T_2}\right)^{1/2}.$$

Shock Mach number – 4

- The post-shock Mach number is

$$\mathcal{M}_2 \equiv \frac{v_2}{c_2} = \frac{v_1}{c_1}\frac{v_2}{v_1}\frac{c_1}{c_2} = \mathcal{M}_1 \frac{v_2}{v_1}\left(\frac{T_1}{T_2}\right)^{1/2}.$$

- This simplifies in the strong-shock limit, yielding

$$\mathcal{M}_2 \approx \mathcal{M}_1 \frac{\gamma-1}{\gamma+1}\left[\frac{(\gamma+1)^2}{2\gamma(\gamma-1)\mathcal{M}_1^2}\right]^{1/2} = \left(\frac{\gamma-1}{2\gamma}\right)^{1/2} \approx 0.45.$$

- A shock converts supersonic gas into denser, slower moving, higher pressure, subsonic gas because (for strong shocks and $\gamma = 5/3$)

$$\frac{\rho_2}{\rho_1} = 4,$$
$$\frac{v_2}{v_1} = \frac{1}{4},$$
$$\frac{P_2}{P_1} = \frac{5}{4}\mathcal{M}_1^2 > 1,$$
$$\mathcal{M}_2 \approx 0.45.$$

Shock adiabatic curve – 1

- The shock increases the specific entropy of the gas by an amount

$$s_2 - s_1 = c_V \ln\left(\frac{P_2}{\rho_2^\gamma}\right) - c_V \ln\left(\frac{P_1}{\rho_1^\gamma}\right)$$
$$= c_V \ln\left(\frac{P_2}{P_1}\right) - c_V\gamma \ln\left(\frac{\rho_2}{\rho_1}\right) = c_V \ln\left(\frac{K_2}{K_1}\right).$$

Hence, the shock shifts the gas to a higher adiabatic curve that is uniquely labeled by $K = P\rho^{-\gamma}$: gas can move adiabatically along an adiabatic curve while changes in entropy move it from one adiabatic curve to another.

Shock adiabatic curve – 1

- The shock increases the specific entropy of the gas by an amount

$$s_2 - s_1 = c_V \ln\left(\frac{P_2}{\rho_2^\gamma}\right) - c_V \ln\left(\frac{P_1}{\rho_1^\gamma}\right)$$
$$= c_V \ln\left(\frac{P_2}{P_1}\right) - c_V\gamma \ln\left(\frac{\rho_2}{\rho_1}\right) = c_V \ln\left(\frac{K_2}{K_1}\right).$$

  Hence, the shock shifts the gas to a higher adiabatic curve that is uniquely labeled by $K = P\rho^{-\gamma}$: gas can move adiabatically along an adiabatic curve while changes in entropy move it from one adiabatic curve to another.

- With the definition of the current density $j = \rho_1 v_1 = \rho_2 v_2 =$ const., we obtain for the 2nd Rankine Hugoniot condition (Eq. 12)

$$[\rho v^2 + P] = \left[\frac{j^2}{\bar{m}} V + P\right] = 0 \implies \frac{j^2}{\bar{m}} V_1 + P_1 = \frac{j^2}{\bar{m}} V_2 + P_2.$$

Shock adiabatic curve – 1

- The shock increases the specific entropy of the gas by an amount

$$
\begin{aligned}
s_2 - s_1 &= c_V \ln\left(\frac{P_2}{\rho_2^\gamma}\right) - c_V \ln\left(\frac{P_1}{\rho_1^\gamma}\right) \\
&= c_V \ln\left(\frac{P_2}{P_1}\right) - c_V \gamma \ln\left(\frac{\rho_2}{\rho_1}\right) = c_V \ln\left(\frac{K_2}{K_1}\right).
\end{aligned}
$$

  Hence, the shock shifts the gas to a higher adiabatic curve that is uniquely labeled by $K = P\rho^{-\gamma}$: gas can move adiabatically along an adiabatic curve while changes in entropy move it from one adiabatic curve to another.

- With the definition of the current density $j = \rho_1 v_1 = \rho_2 v_2 = \text{const.}$, we obtain for the 2$^{\text{nd}}$ Rankine Hugoniot condition (Eq. 12)

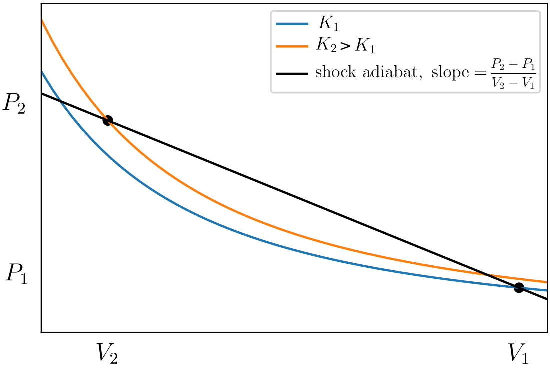
$$
[\rho v^2 + P] = \left[\frac{j^2}{\bar{m}} V + P\right] = 0 \implies \frac{j^2}{\bar{m}} V_1 + P_1 = \frac{j^2}{\bar{m}} V_2 + P_2.
$$

- Hence, the slope of the shock adiabatic curve in the P-V diagram is

$$
-\frac{j^2}{\bar{m}} = -\frac{P_2 - P_1}{V_1 - V_2}.
$$

AIP

Shock adiabatic curve – 2



- The slope of the shock adiabatic curve in the *P*-*V* diagram is

$$-\frac{j^2}{\bar{m}} = -\frac{P_2 - P_1}{V_1 - V_2} = -\frac{\Delta P}{\Delta V},$$

which connects the two adiabatic curves of the up- and down stream fluid.

Oblique shocks – 1

- Now, we generalize our plane-parallel shock geometry to account for the fluid to impact the shock at some oblique angle. We define a velocity component parallel to the shock normal, $v_{||} \equiv \boldsymbol{v} \cdot \boldsymbol{n}$, as well as a perpendicular component, $v_{\perp}$.

AIP

Oblique shocks – 1

- Now, we generalize our plane-parallel shock geometry to account for the fluid to impact the shock at some oblique angle. We define a velocity component parallel to the shock normal, $v_{||} \equiv \boldsymbol{v} \cdot \boldsymbol{n}$, as well as a perpendicular component, $v_{\perp}$.
- The momentum conservation equation

$$\frac{\partial}{\partial t}(\rho \boldsymbol{v}) + \boldsymbol{\nabla} \cdot \left(\rho \boldsymbol{v} \boldsymbol{v}^{\mathsf{T}} + P\bar{\mathbf{1}} - \bar{\boldsymbol{\Pi}} \right) = \mathbf{0}.$$

defines a momentum current through a unit surface area with normal vector $\boldsymbol{n}$ (neglecting viscosity outside the shock transition layer and splitting the flux into a normal and a perpendicular component),

$$\rho \boldsymbol{v}(\boldsymbol{v} \cdot \boldsymbol{n}) + P\boldsymbol{n}.$$

Oblique shocks – 1

- Now, we generalize our plane-parallel shock geometry to account for the fluid to impact the shock at some oblique angle. We define a velocity component parallel to the shock normal, $v_{||} \equiv \boldsymbol{v} \cdot \boldsymbol{n}$, as well as a perpendicular component, $v_{\perp}$.
- The momentum conservation equation

$$\frac{\partial}{\partial t}(\rho \boldsymbol{v}) + \boldsymbol{\nabla} \cdot \left(\rho \boldsymbol{v}\boldsymbol{v}^{\mathsf{T}} + P\bar{\mathbf{1}} - \bar{\boldsymbol{\Pi}}\right) = \mathbf{0}.$$

defines a momentum current through a unit surface area with normal vector $\boldsymbol{n}$ (neglecting viscosity outside the shock transition layer and splitting the flux into a normal and a perpendicular component),

$$\rho \boldsymbol{v}(\boldsymbol{v} \cdot \boldsymbol{n}) + P\boldsymbol{n}.$$

- The momentum current has to be continuous across the shock in order for the forces that are acting on both sides of the shock on the gas to be identical. In our case, $\boldsymbol{n}$ coincides with the shock normal and points along $\boldsymbol{e}_x$. Continuity of the x, y, and z components of the momentum current yields

$$[\rho v_x^2 + P] = 0,$$
$$[\rho v_x v_y] = 0,$$
$$[\rho v_x v_z] = 0.$$

Oblique shocks – 2

- We recap the last set of equations:

$$
\begin{aligned}
[\rho v_x^2 + P] &= 0, \\
[\rho v_x v_y] &= 0, \\
[\rho v_x v_z] &= 0.
\end{aligned}
$$

Oblique shocks – 2

- We recap the last set of equations:

$$[\rho v_x^2 + P] = 0,$$
$$[\rho v_x v_y] = 0,$$
$$[\rho v_x v_z] = 0.$$

- At a shock $j = \rho v_x \neq 0$ and $\rho \neq 0$ so that we get

$$[v_y] = 0 \quad \text{and} \quad [v_z] = 0,$$

i.e., the tangential velocities are continuous across the shock. Thus, only the parallel velocity component is modified at a shock according to $v_{\parallel,2} = v_{\parallel,1}\rho_1/\rho_2$ while the perpendicular component remains invariant, $v_{\perp,1} = v_{\perp,2} = v_{\perp}$.

AIP

Oblique shocks – 2

- We recap the last set of equations:

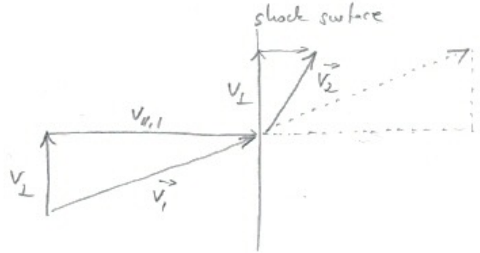
$$[\rho v_x^2 + P] = 0,$$
$$[\rho v_x v_y] = 0,$$
$$[\rho v_x v_z] = 0.$$

- At a shock $j = \rho v_x \neq 0$ and $\rho \neq 0$ so that we get

$$[v_y] = 0 \quad \text{and} \quad [v_z] = 0,$$

i.e., the tangential velocities are continuous across the shock. Thus, only the parallel velocity component is modified at a shock according to $v_{\parallel,2} = v_{\parallel,1}\rho_1/\rho_2$ while the perpendicular component remains invariant, $v_{\perp,1} = v_{\perp,2} = v_{\perp}$.

- This implies a refraction of the (oblique) flow toward the shock surface.



Oblique shocks – 3

- If the flow impinges with a constant angle at a shock, it is deflected by the same amount everywhere along the shock surface.

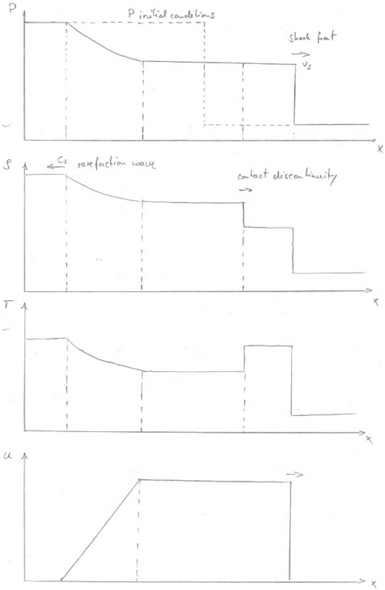
Oblique shocks – 3

- If the flow impinges with a constant angle at a shock, it is deflected by the same amount everywhere along the shock surface.
- Consider now a curved shock: this implies a changing angle between $\boldsymbol{v}$ and $\boldsymbol{n}$ along the shock surface and hence, the shock transition causes a different amount of "shock deflection" of the velocity field.
- As a result, there is shear injected at a shock because two infinitesimally separated points on the shock surface experience a different amount of deflection.

Oblique shocks – 3

- If the flow impinges with a constant angle at a shock, it is deflected by the same amount everywhere along the shock surface.
- Consider now a curved shock: this implies a changing angle between $\boldsymbol{v}$ and $\boldsymbol{n}$ along the shock surface and hence, the shock transition causes a different amount of “shock deflection” of the velocity field.
- As a result, there is shear injected at a shock because two infinitesimally separated points on the shock surface experience a different amount of deflection.
- This implies subsonic (solenoidal) turbulence in the post-shock regime: the vorticity is injected at the curvature radius of the shock and cascades down in scale to the Kolmogorov scale where it gets dissipated. Hence, in a curved shock, there is eventually more kinetic energy dissipated into heat in comparison to an oblique shock without curvature that experiences the same amount of ram pressure.

Riemann problem



Generalized Rankine-Hugoniot conditions

- In the exercises, you derived the generalized Rankine-Hugoniot shock jump conditions of mass, momentum, and energy conservation,

$$
\begin{aligned}
v_{\mathrm{s}}[\rho] &= [\rho u], \\
v_{\mathrm{s}}[\rho u] &= [\rho u^2 + P], \\
v_{\mathrm{s}}\left[\rho \frac{u^2}{2} + \varepsilon\right] &= \left[\left(\rho \frac{u^2}{2} + \varepsilon + P\right) u\right].
\end{aligned}
$$

Here v_{s} and u denote the shock and the mean gas velocity measured in the laboratory rest system.

AIP

Generalized Rankine-Hugoniot conditions

- In the exercises, you derived the generalized Rankine-Hugoniot shock jump conditions of mass, momentum, and energy conservation,

$$\begin{aligned} v_s[\rho] &= [\rho u], \\ v_s[\rho u] &= [\rho u^2 + P], \\ v_s\left[\rho\frac{u^2}{2} + \varepsilon\right] &= \left[\left(\rho\frac{u^2}{2} + \varepsilon + P\right) u\right]. \end{aligned}$$

  Here v_s and u denote the shock and the mean gas velocity measured in the laboratory rest system.

- Setting $v_s = 0$ transforms us to the rest system of the shock. These generalized Rankine-Hugoniot conditions are very useful for deriving the analytic solution of the Riemann problem.

AIP

- **General properties of shocks**
  * shocks are produced via sound wave steepening or via a supersonic compressible disturbance (non-linearity of Navier Stokes equations)
  * shocks: sudden change in density, temperature, and pressure that decelerates a supersonic flow
  * shocks are ubiquitous: cluster mergers, AGN jets, supernova explosions
  * **Rankine-Hugoniot jump conditions** at the shock manifest conservation laws: ρ jumps by 4 ($\gamma = 5/3$) at a strong shock, T and P are not bounded

- **General properties of shocks**
  - * shocks are produced via sound wave steepening or via a supersonic compressible disturbance (non-linearity of Navier Stokes equations)
  - * shocks: sudden change in density, temperature, and pressure that decelerates a supersonic flow
  - * shocks are ubiquitous: cluster mergers, AGN jets, supernova explosions
  - * **Rankine-Hugoniot jump conditions** at the shock manifest conservation laws: ρ jumps by 4 ($\gamma = 5/3$) at a strong shock, T and P are not bounded
- **Tangential discontinuities and curved shocks**
  - * ***tangential discontinuity:*** P, v and mass flux ρv are all constant across
  - * ***oblique shock:*** deflects the (oblique) flow toward the shock surface
  - * ***curved shock:*** deflection angle changes along shock surface: generation of shear and injection of vorticity

- **General properties of shocks**
  - * shocks are produced via sound wave steepening or via a supersonic compressible disturbance (non-linearity of Navier Stokes equations)
  - * shocks: sudden change in density, temperature, and pressure that decelerates a supersonic flow
  - * shocks are ubiquitous: cluster mergers, AGN jets, supernova explosions
  - * **Rankine-Hugoniot jump conditions** at the shock manifest conservation laws: ρ jumps by 4 ($\gamma = 5/3$) at a strong shock, T and P are not bounded
- **Tangential discontinuities and curved shocks**
  - * ***tangential discontinuity:*** P, v and mass flux ρv are all constant across
  - * ***oblique shock:*** deflects the (oblique) flow toward the shock surface
  - * ***curved shock:*** deflection angle changes along shock surface: generation of shear and injection of vorticity
- **Astrophysical collisionless shocks** can
  - * accelerate particles (electrons and ions) $\rightarrow$ cosmic rays (CRs)
  - * amplify magnetic fields (or generate them from scratch)
  - * exchange energy between electrons and ions

AIP