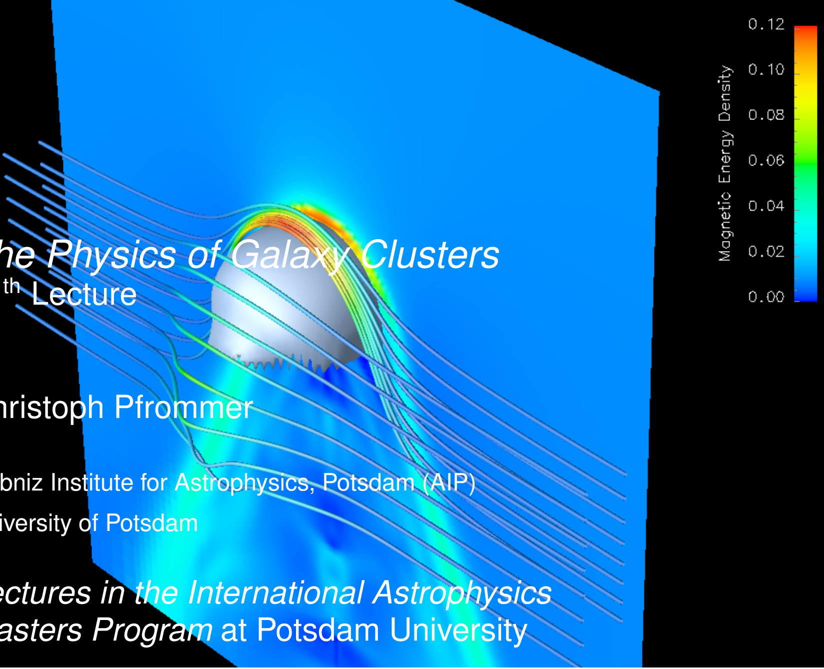


The Physics of Galaxy Clusters

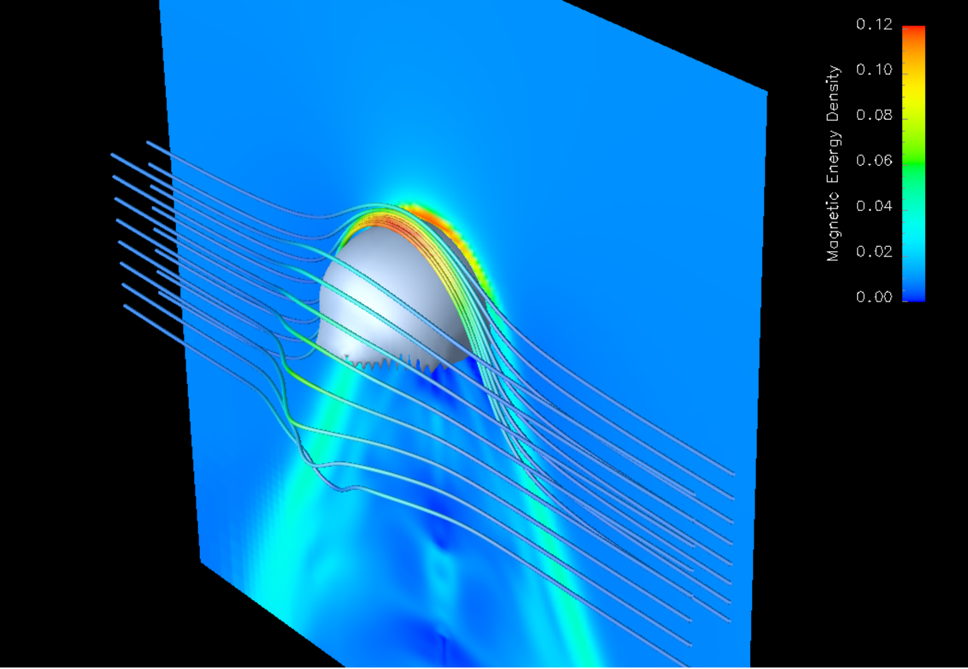
11^{th} Lecture

Christoph Pfrommer

Leibniz Institute for Astrophysics, Potsdam (AIP)

University of Potsdam

Lectures in the International Astrophysics Masters Program at Potsdam University


Magnetic Energy Density
0.12
0.10
0.08
0.06
0.04
0.02
0.00

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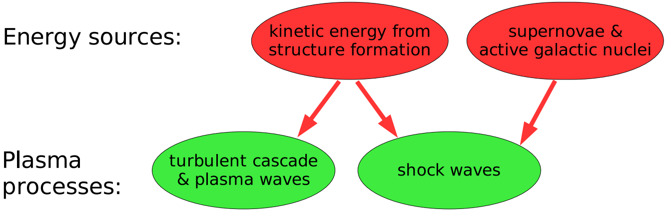
- **Thermal instability with magnetic fields:**
  - * magnetic fields fundamentally change the convective stability criteria of weakly collisional plasmas such as the ICM
  - * CC cluster centers are subject to the heat-flux driven buoyancy instability
  - * the outskirts of all clusters are subject to the magneto-thermal instability

Multi messenger approach for non-thermal processes

Relativistic populations and radiative processes in clusters:



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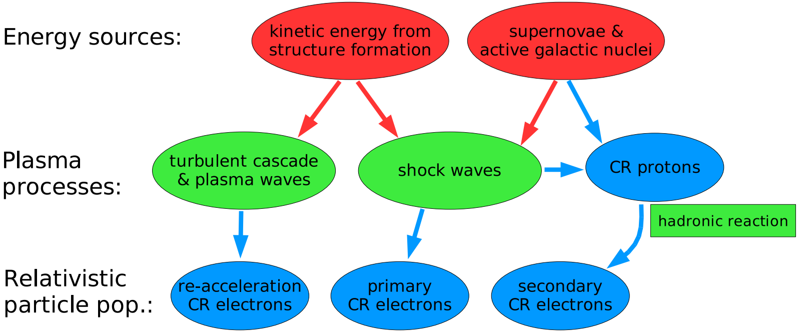
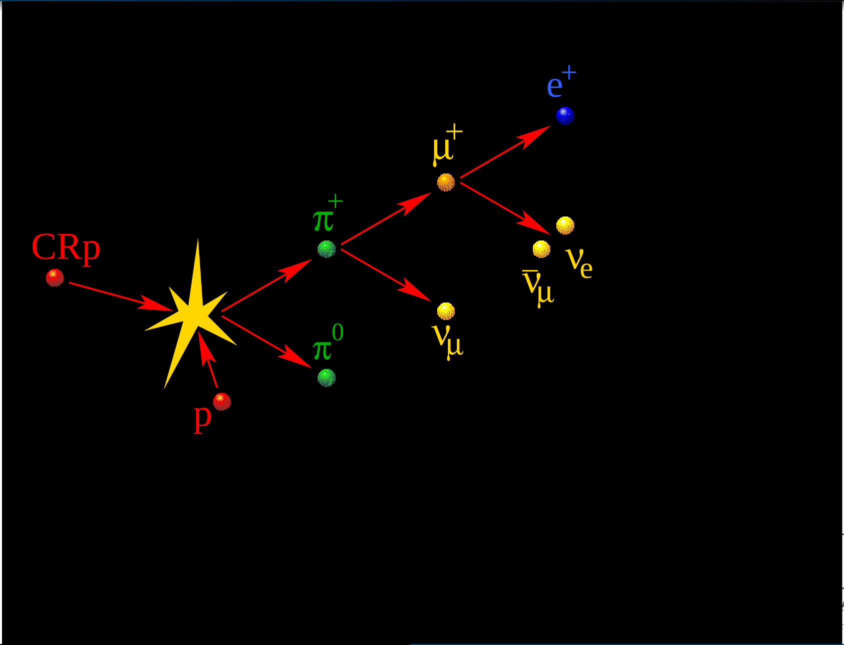


Hadronic cosmic ray proton interaction



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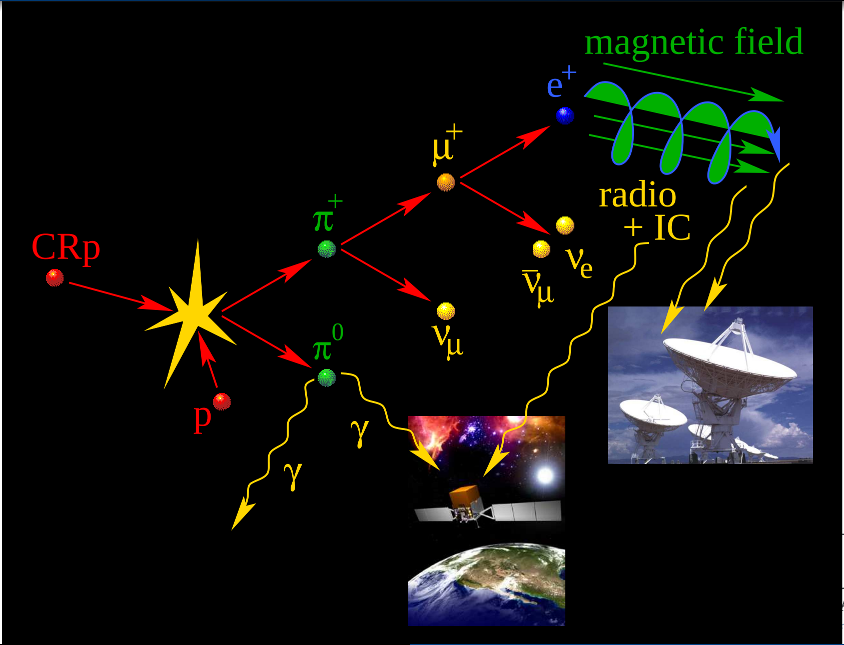

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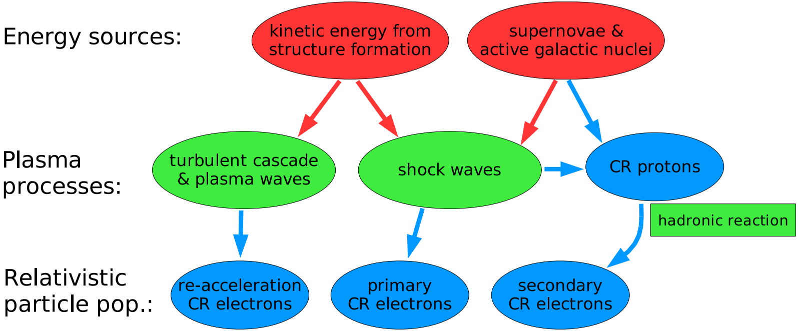

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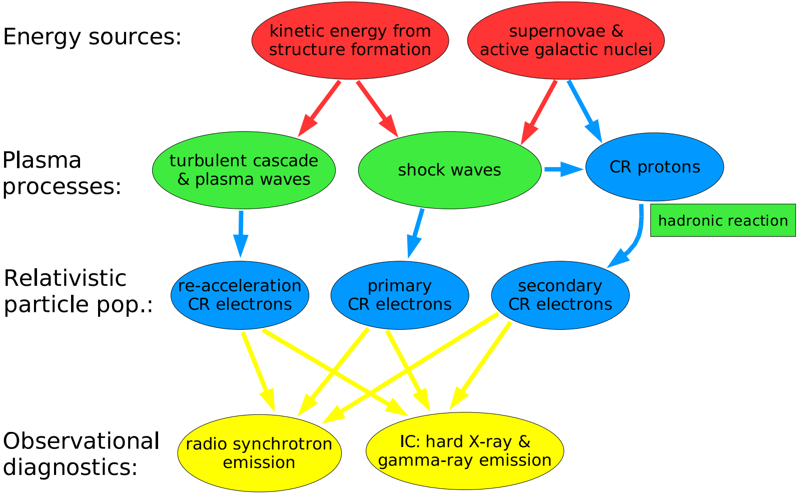

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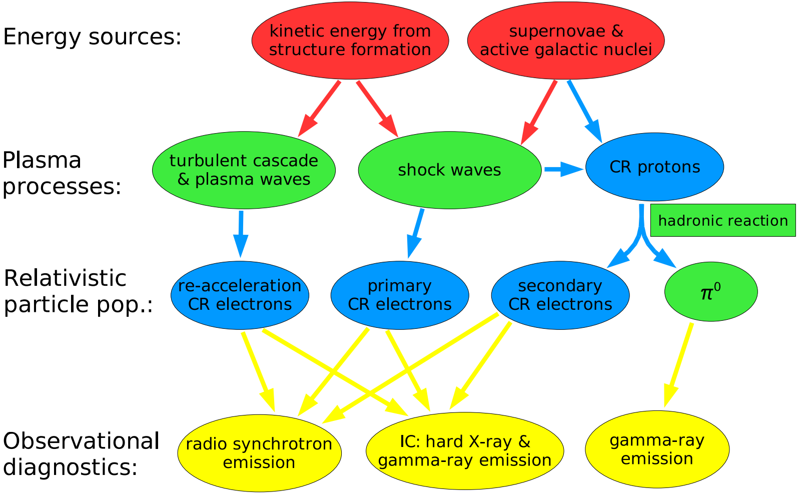
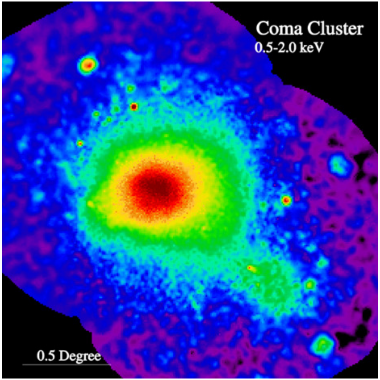


What we hope to learn from non-thermal emission

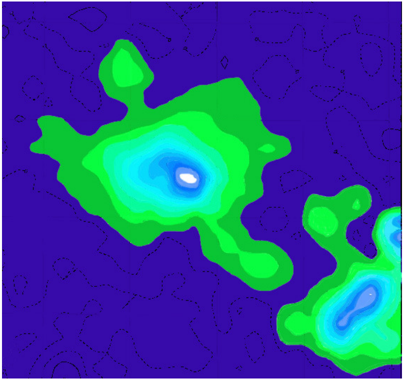
- understanding plasma astrophysics in high-beta plasmas:
  - shock and particle acceleration
  - large-scale magnetic fields
  - turbulence
- dynamical state $\rightarrow$ cluster cosmology?
  - non-thermal pressure support: hydrostatics + Sunyaev-Zel'dovich effect
  - history of individual clusters: cluster archaeology
  - illuminating the process of structure formation
- consistent picture of non-thermal processes:
  radio, soft/hard X-rays, γ-rays

Giant radio halo in the Coma cluster



thermal X-ray emission

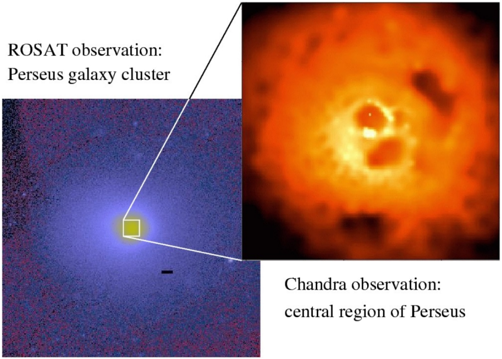
(Snowden/MPE/ROSAT)



radio synchrotron emission

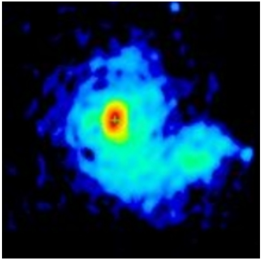
(Deiss/Effelsberg)

Radio mini halo in the Perseus cluster



thermal X-ray emission

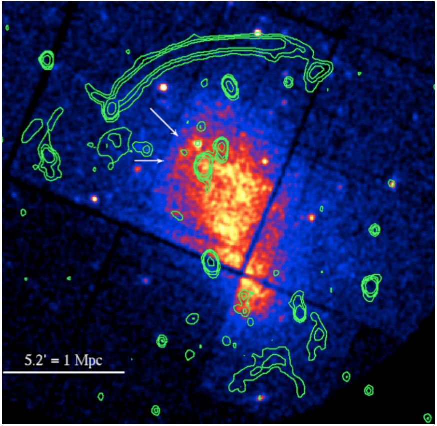
(ROSAT; NASA/IoA/A.Fabian et al.)



radio synchrotron emission

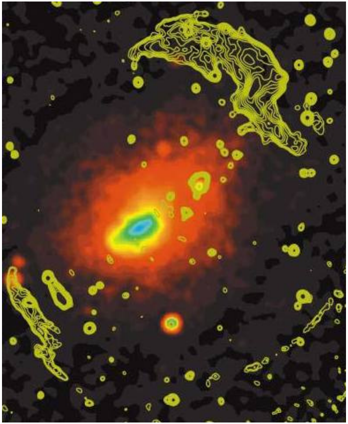
(Pedlar/VLA)

Giant radio relics in merging clusters



CIZA J2242.8+5301 ("sausage relic")

(X-ray: XMM; radio: WSRT; Ogrean+ 2013)



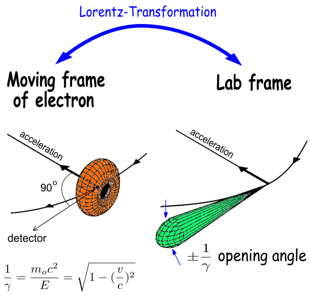
Abell 3667

(radio: Johnston-Hollitt. X-ray: ROSAT/PSPC.)

Observing magnetic fields in clusters – 1

Synchrotron emission:

- Charged particles emit electromagnetic radiation when accelerated, e.g. due to the Lorentz force of a magnetic field.
- This emission is axisymmetric with respect to the acceleration direction in the *particle's rest frame*.
- If the particles move relativistically, then the emission in the *lab frame* is beamed into a forward cone of an opening angle $\theta \sim \gamma^{-1}$ (where γ is the Lorentz factor).



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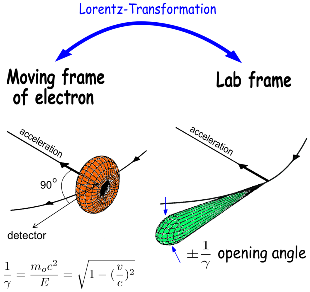
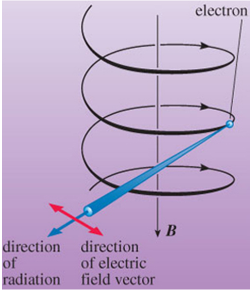
- Because the emission (= transverse electromagnetic wave) propagates in a narrow cone, it is linearly polarized.
- The typical synchrotron frequency is

$$\nu_{\mathrm{synch}} = \frac{3eB}{2\pi\, m_e c}\,\gamma^2 \simeq 1\ \mathrm{GHz}\ \frac{B}{\mu\mathrm{G}} \left(\frac{\gamma}{10^4}\right)^2 .$$

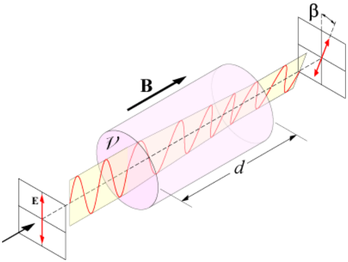
- Power-law cosmic ray electron momentum distributions imply power-law (radio) synchrotron spectra.





Faraday rotation:

- Faraday rotation describes rotation of a linearly polarized electro-magnetic wave in the presence of a line-of-sight (LOS) magnetic field because of the birefringent property of a plasma.
- This can be seen by splitting the linearly polarized wave into right- and left-hand circularly polarized waves, which propagate at slightly different speeds.
- The observed polarization angle ϕ_{obs} is modified from its intrinsic position angle, $\phi_{\mathrm{intrinsic}}$.



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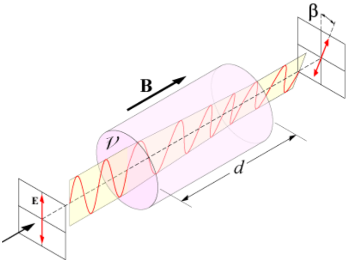
$$\phi_{\mathrm{obs}}(\boldsymbol{x}_\perp) = \lambda^2 \mathrm{RM}(\boldsymbol{x}_\perp) + \phi_{\mathrm{intrinsic}}(\boldsymbol{x}_\perp),$$

$$\mathrm{RM}(\boldsymbol{x}_\perp) = \frac{e^3}{2\pi\, m_{\mathrm{e}}^2 c^4} \int_0^d n_{\mathrm{e}}(\boldsymbol{x}_\perp, l)\, \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l}$$

$$= 812\, \frac{\mathrm{rad}}{\mathrm{m}^2}\, \frac{B}{\mu\mathrm{G}}\, \frac{n_{\mathrm{e}}}{10^{-3}\mathrm{cm}^{-3}}\, \frac{d}{\mathrm{Mpc}}.$$



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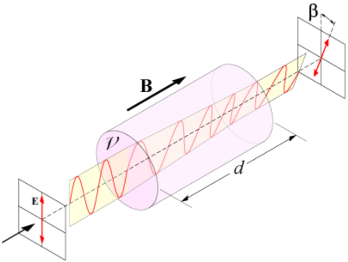

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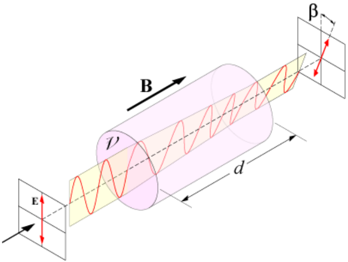

- If you decrease the magnetic coherence length while leaving the gas density and magnetic field strength invariant, what happens to the value of RM?

AIP

Observing magnetic fields in clusters – 3

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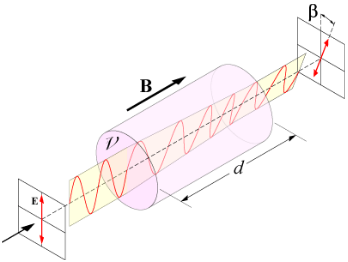

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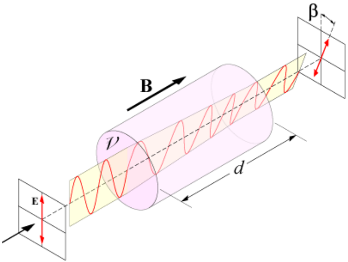

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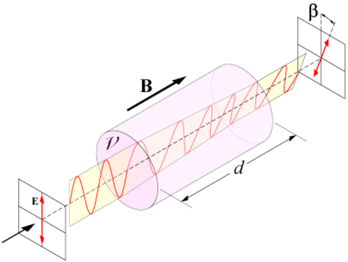

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- Explain the phenomenon of the $n\pi$ ambiguity for the observable polarization angle. What could you do to circumvent it?
  $\Rightarrow$ ***measure the RM signal more finely spaced in frequency*** within a given radio band

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Origin and growth of magnetic fields

The general picture:

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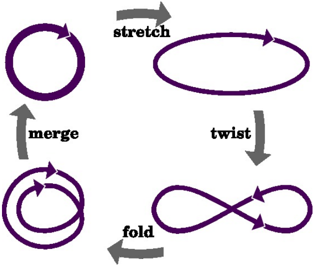

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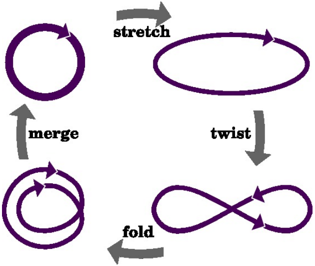
- ***Growth.*** A small-scale (fluctuating) dynamo is an MHD process, in which the kinetic (turbulent) energy is converted into magnetic energy: the mechanism relies on magnetic fields to become stronger when the field lines are stretched
- ***Saturation.*** Field growth stops at a sizeable fraction of the turbulent energy when magnetic forces become strong enough to resist the stretching and folding motions



The Biermann battery – 1

- Electron and proton momenta change due to the Lorentz force, the pressure and viscous forces:

$$m_\mathrm{e}\frac{\mathrm{d}\boldsymbol{v}_\mathrm{e}}{\mathrm{d}t} = -e\left(\boldsymbol{E} + \frac{\boldsymbol{v}_\mathrm{e}}{c}\times\boldsymbol{B} + \frac{1}{en_\mathrm{e}}\boldsymbol{\nabla}P_\mathrm{e}\right) - \frac{\nu_\mathrm{visc}m_\mathrm{e}}{n_\mathrm{e}}(\boldsymbol{v}_\mathrm{e} - \boldsymbol{v}_\mathrm{p}),$$

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- If $T_\mathrm{p} = T_\mathrm{e}$, we can neglect the proton equation because protons move on average slower than electrons by a factor $\sqrt{m_\mathrm{p}/m_\mathrm{e}}$.
- Viscous forces are very small on large scales: we drop the term $\propto \nu_\mathrm{visc}$.
- We assume a steady state (i.e., $\tau > \omega_\mathrm{pl}^{-1}$, where $\omega_\mathrm{pl}^2 = 4\pi n_\mathrm{e}e^2/m_\mathrm{e}$ is the plasma frequency) and solve for $\boldsymbol{E}$:

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$$\boldsymbol{E} = -\frac{\boldsymbol{v}_\mathrm{e}\times\boldsymbol{B}}{c}-\frac{\boldsymbol{\nabla}P_\mathrm{e}}{en_\mathrm{e}}.$$

- Multiplying this equation by $-c$, taking the curl of it and using Faraday's law, we obtain

$$\frac{\partial\boldsymbol{B}}{\partial t} = -c\boldsymbol{\nabla}\times\boldsymbol{E} = \boldsymbol{\nabla}\times(\boldsymbol{v}_\mathrm{e}\times\boldsymbol{B})+\frac{c}{e}\boldsymbol{\nabla}\times\left(\frac{\boldsymbol{\nabla}P_\mathrm{e}}{n_\mathrm{e}}\right). \tag{1}$$

The Biermann battery – 2

- Using $P_\mathrm{e} = n_\mathrm{e} k_\mathrm{B} T_\mathrm{e}$ and the identities $\boldsymbol{\nabla} \times (f \boldsymbol{\nabla} g) \equiv \boldsymbol{\nabla} f \times \boldsymbol{\nabla} g$ and $\boldsymbol{\nabla} \times \boldsymbol{\nabla} f \equiv \mathbf{0}$, we can rewrite the second term of Eq. (1):

$$
\begin{aligned}
\frac{1}{k_\mathrm{B}} \boldsymbol{\nabla} \times \left(\frac{\boldsymbol{\nabla} P_\mathrm{e}}{n_\mathrm{e}} \right) &= \boldsymbol{\nabla} \times \left[\frac{1}{n_\mathrm{e}} \boldsymbol{\nabla} (n_\mathrm{e} T_\mathrm{e}) \right] = \boldsymbol{\nabla} \times (\boldsymbol{\nabla} T_\mathrm{e}) + \boldsymbol{\nabla} \times \left(\frac{T_\mathrm{e}}{n_\mathrm{e}} \boldsymbol{\nabla} n_\mathrm{e} \right) \\
&= \boldsymbol{\nabla} \left(\frac{T_\mathrm{e}}{n_\mathrm{e}} \right) \times \boldsymbol{\nabla} n_\mathrm{e} = \frac{1}{n_\mathrm{e}} \boldsymbol{\nabla} T_\mathrm{e} \times \boldsymbol{\nabla} n_\mathrm{e} - \frac{T_\mathrm{e}}{n_\mathrm{e}^2} \boldsymbol{\nabla} n_\mathrm{e} \times \boldsymbol{\nabla} n_\mathrm{e} \\
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&= \frac{1}{n_\mathrm{e}} \boldsymbol{\nabla} T_\mathrm{e} \times \boldsymbol{\nabla} n_\mathrm{e}.
\end{aligned}
$$

- Hence, we obtain the Biermann battery equation,

$$
\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{v}_\mathrm{e} \times \boldsymbol{B}) - \frac{c k_\mathrm{B}}{e n_\mathrm{e}} \boldsymbol{\nabla} n_\mathrm{e} \times \boldsymbol{\nabla} T_\mathrm{e}.
$$

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The Biermann battery – 2

- Using $P_\mathrm{e} = n_\mathrm{e} k_\mathrm{B} T_\mathrm{e}$ and the identities $\boldsymbol{\nabla} \times (f \boldsymbol{\nabla} g) \equiv \boldsymbol{\nabla} f \times \boldsymbol{\nabla} g$ and $\boldsymbol{\nabla} \times \boldsymbol{\nabla} f \equiv \mathbf{0}$, we can rewrite the second term of Eq. (1):

$$\begin{aligned}
\frac{1}{k_\mathrm{B}} \boldsymbol{\nabla} \times \left(\frac{\boldsymbol{\nabla} P_\mathrm{e}}{n_\mathrm{e}} \right) &= \boldsymbol{\nabla} \times \left[\frac{1}{n_\mathrm{e}} \boldsymbol{\nabla} (n_\mathrm{e} T_\mathrm{e}) \right] = \boldsymbol{\nabla} \times (\boldsymbol{\nabla} T_\mathrm{e}) + \boldsymbol{\nabla} \times \left(\frac{T_\mathrm{e}}{n_\mathrm{e}} \boldsymbol{\nabla} n_\mathrm{e} \right) \\
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- This equation shows that if there is no magnetic field to start with (i.e., a vanishing first term on the right-hand side), then the magnetic field can be generated by a baroclinic flow with $\boldsymbol{\nabla} n_\mathrm{e} \times \boldsymbol{\nabla} T_\mathrm{e} \neq \mathbf{0}$.

AIP

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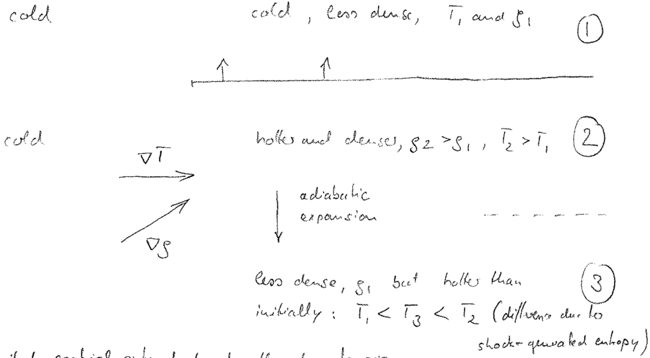
- This could be achieved in shocks of the interstellar medium, in ionization fronts, or similar astrophysical sites; in general, baroclinic flows are sourced by rotational motions at shocks of finite extent such as the chaotic collapse of a proto-galaxy.

The Biermann battery – 3

- Consider a shock of finite extent that propagates into zero-pressure medium.



$\Rightarrow$ shocks of limited spatial extend break the barotropic relation $P_e = P_e(n_e)$ which couples P_e to n_e

The Biermann battery – 4

- Magnetic fields generated through this process have very small field strengths: adopting a characteristic density and temperature gradient length of L of a proto-galaxy and assuming gravitational collapse on the free-fall time, $\tau \sim 1/\sqrt{G\rho}$, we obtain

$$B \sim \frac{ck_\mathrm{B}T_\mathrm{e}}{e}\,\frac{\tau}{L^2} \sim \frac{ck_\mathrm{B}T_\mathrm{e}}{e}\,\frac{1}{\sqrt{G\rho}L^2}$$
$$\sim 10^{-20}\mathrm{G}\left(\frac{T_\mathrm{e}}{10^4\,\mathrm{K}}\right)\left(\frac{n}{1\,\mathrm{cm}^{-3}}\right)^{-1/2}\left(\frac{L}{\mathrm{kpc}}\right)^{-2}.$$

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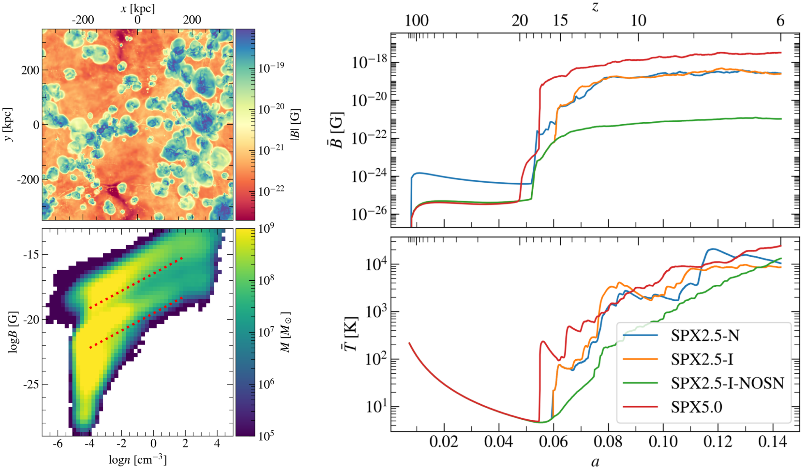
- Naively, going to smaller length scales L should increase B. However, in order to explain the *coherence on scales of several kpcs*, we would have to evoke a process such as a small-scale wind that moves the magnetic fields back to kpc scales and in that process we would have to account for adiabatic losses that accompany the expansion from small to large scales: in the end we would gain nothing from running a Biermann battery on smaller scales.
- This solves the cosmological magneto-genesis problem, but the big challenge remains in growing coherent large-scale magnetic fields from a small-amplitude, small-scale fields: this is a major challenge of dynamo theory!

AIP

Cosmological magneto-genesis: the Biermann battery



- Cosmological simulations of the Biermann battery during the epoch of reionization with a state-of-the-art galaxy formation model find magnetic field generation at reionization fronts and at supernova-driven outflows (Attia+ 2021)

The induction equation – 1

- To derive the equations of magneto-hydrodynamics, we need
  1. an evolution equation of the magnetic field embedded in a fluid flow, and
  2. work out the magnetic force and stress.

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- If the fluid moves with the velocity $\boldsymbol{v}$ relative to the observer, we have the component of the electric field parallel to the flow, $E_{\|} = \boldsymbol{E} \cdot \boldsymbol{v}/v$, and perpendicular to the flow, $\boldsymbol{E}_{\perp} = \boldsymbol{E} - E_{\|}\boldsymbol{v}/v$. The Lorentz transformation into observer's frame (unprimed quantities) is given by

$$E'_{\|} = E_{\|},$$
$$\boldsymbol{E}'_{\perp} = \gamma \left(\boldsymbol{E}_{\perp} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B} \right),$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$ is the Lorentz factor.

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  where $\gamma = (1 - v^2/c^2)^{-1/2}$ is the Lorentz factor.
- In the non-relativistic limit ($\boldsymbol{j}' = \boldsymbol{j}$ and $\gamma = 1$), we have Ohm's law in the observer's frame:

$$\boldsymbol{E} = -\frac{\boldsymbol{v}}{c} \times \boldsymbol{B} + \eta \boldsymbol{j}.$$

The induction equation – 2

- Let's recall Ohm's law:

$$\boldsymbol{E} = -\frac{\boldsymbol{v}}{c} \times \boldsymbol{B} + \eta\,\boldsymbol{j},$$

- Using Faraday's Law, $\dfrac{\partial \boldsymbol{B}}{\partial t} = -c\,\boldsymbol{\nabla} \times \boldsymbol{E}$, we get

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B}) - \boldsymbol{\nabla} \times (c\,\eta\,\boldsymbol{j}).$$

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$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B}) + D\boldsymbol{\nabla}^2\boldsymbol{B}, \quad \text{where} \quad D = \frac{c^2\,\eta}{4\pi}.$$

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- At small frequencies (when the displacement current in Ampère's law is negligible), the induction equation has changed character: from Faraday's law describing the generation of voltages by changing magnetic fields in coils, we found an evolution equation of the magnetic field embedded in a fluid flow.

The induction equation – discussion

- The magnetic induction equation reads:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B}) + D\boldsymbol{\nabla}^2\boldsymbol{B}, \quad \text{where} \quad D = \frac{c^2\,\eta}{4\pi}. \tag{2}$$

- 1^{st} term: the "convective term" states that the field is frozen into the flow (as we will see momentarily): an important property for astrophysical plasmas!
- 2^{nd} term: the "diffusive term" represents the diffusive leakage of magnetic field lines across the conducting field, which is important for changing the magnetic topology, e.g. in reconnection.

AIP

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- The "diffusive term" can be neglected for infinite conductivity $\sigma = \eta^{-1}$ or for large magnetic Reynolds numbers $\mathrm{Re}_{\mathrm{m}} \to \infty$:

$$\mathrm{Re}_{\mathrm{m}} = \frac{|\text{convective term}|}{|\text{diffusive term}|} = \frac{L^{-1} v\, B}{D\, L^{-2} B} = \frac{L\, v}{D}$$

AIP

Magnetic force and stress – 1

Using Ampère's law at low frequencies, $\boldsymbol{\nabla} \times \boldsymbol{B} = 4\pi \boldsymbol{j}/c$, we will now show that the Lorentz force density can be written as follows:

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where

$$\mathrm{M}_{ij} = -\frac{1}{4\pi} B_i B_j + \frac{1}{8\pi} B^2 \delta_{ij}$$

is the magnetic stress tensor: it plays a role analogous to the fluid pressure in ordinary fluid mechanics (explaining the minus sign introduced in its definition).

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- We have

$$\begin{aligned}(\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B}|_i &= \varepsilon_{ijk}\varepsilon_{jlm}(\partial_l B_m) B_k = \varepsilon_{kij}\varepsilon_{jlm}(\partial_l B_m) B_k \\ &= (\delta_{kl}\delta_{im} - \delta_{km}\delta_{il})(\partial_l B_m) B_k = (\partial_k B_i) B_k - (\partial_i B_k) B_k \\ &= \left[(\boldsymbol{B} \cdot \boldsymbol{\nabla})\boldsymbol{B} - \frac{1}{2}\boldsymbol{\nabla}\boldsymbol{B}^2\right]_i .\end{aligned}$$

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- In this notation, the magnetic *curvature force* and the *magnetic pressure* force are not separated.

Magnetic force and stress – 2

- In order to fully separate the effects of magnetic curvature and pressure we write $\boldsymbol{B} = B\boldsymbol{b}$, where $\boldsymbol{b}$ is the unit vector in the direction of $\boldsymbol{B}$ and obtain

$$
\begin{aligned}
\boldsymbol{f}_{\mathrm{L}} &= \frac{1}{4\pi}\left(\boldsymbol{B}\cdot\boldsymbol{\nabla}\right)\boldsymbol{B} - \frac{1}{8\pi}\boldsymbol{\nabla}B^2 \\
&= \frac{B^2}{4\pi}\left(\boldsymbol{b}\cdot\boldsymbol{\nabla}\right)\boldsymbol{b} + \frac{1}{8\pi}\boldsymbol{b}(\boldsymbol{b}\cdot\boldsymbol{\nabla})B^2 - \frac{1}{8\pi}\boldsymbol{\nabla}B^2 \\
&= \frac{B^2}{4\pi}\left(\boldsymbol{b}\cdot\boldsymbol{\nabla}\right)\boldsymbol{b} - \frac{1}{8\pi}\boldsymbol{\nabla}_{\perp}B^2 \equiv \boldsymbol{f}_{\mathrm{c}} + \boldsymbol{f}_{\mathrm{p}},
\end{aligned}
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  where we define the gradient perpendicular to the magnetic field lines, $\boldsymbol{\nabla}_{\perp} = (\boldsymbol{1} - \boldsymbol{b}\boldsymbol{b})\cdot\boldsymbol{\nabla}$.

AIP

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- The second term, $\boldsymbol{f}_{\mathrm{p}}$, acts like a *pressure force perpendicular to the magnetic field lines* and the first term, $\boldsymbol{f}_{\mathrm{c}}$, is the *magnetic curvature force* that also acts in a plane orthogonal to the field line.

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- To see this, we locally identify a curved field line with its curvature circle so that we can locally define an azimuthally directed field $\boldsymbol{B} = B\boldsymbol{e}_{\varphi}$ in cylindrical coordinates (R, φ, z). Hence, in this case we obtain

$$(\boldsymbol{b}\cdot\boldsymbol{\nabla})\boldsymbol{b} = (\boldsymbol{e}_{\varphi}\cdot\boldsymbol{\nabla})\boldsymbol{e}_{\varphi} = -\frac{\boldsymbol{e}_R}{R}$$

  so that the curvature force always points towards the center of the curvature circle and aims to reduce the curvature by pulling the field line straight with a force that is the greater the smaller the curvature radius is.

Magnetic force and stress – 3

To get a better understanding, we show that the surface force (per unit area) exerted *by* a bounded volume V *on* its surroundings is given by

$$\boldsymbol{f}_S = \boldsymbol{n} \cdot \mathbf{M} = -\frac{1}{4\pi} \boldsymbol{B} B_n + \frac{1}{8\pi} B^2 \boldsymbol{n},$$

where $B_n = \boldsymbol{B} \cdot \boldsymbol{n}$ is the component of $\boldsymbol{B}$ along the outward normal $\boldsymbol{n}$ to the surface of the volume.

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- The net Lorentz force acting *on* a volume V of fluid can be written as an integral of a magnetic stress vector acting on its surface,

$$\int_V \boldsymbol{f}_\mathrm{L}\,\mathrm{d}V = \int_V \frac{1}{4\pi}(\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B}\,\mathrm{d}V = -\int_V \boldsymbol{\nabla} \cdot \mathbf{M}\,\mathrm{d}V = -\oint_S \boldsymbol{n} \cdot \mathbf{M}\,\mathrm{d}S.$$

AIP

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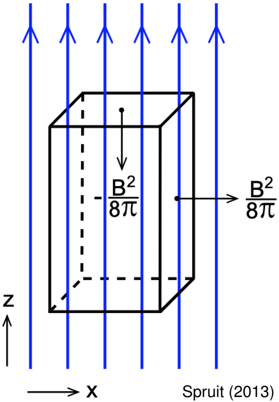
- To get the force $\boldsymbol{f}_S$ exerted *by* the volume *on* its surroundings, we need to add a minus sign to the last term,

$$\boldsymbol{f}_S = \boldsymbol{n} \cdot \mathbf{M} = -\frac{1}{4\pi} \boldsymbol{B} B_n + \frac{1}{8\pi} B^2 \boldsymbol{n},$$

where $B_n = \boldsymbol{B} \cdot \boldsymbol{n}$.

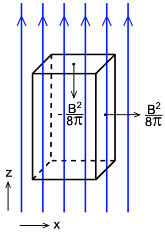
Magnetic force and stress – 4



Spruit (2013)

- We take a uniform magnetic field ($\boldsymbol{B} = B\boldsymbol{e}_z$) and compute the surface forces $\boldsymbol{f}_S$ exerted by a rectangular volume that is aligned with the magnetic field.
- Symmetry limits the surface forces to two different types: 4 with a normal perpendicular to $\boldsymbol{B}$ and 2 with a normal that is (anti-)parallel to $\boldsymbol{B}$.
- Which magnetic forces (pressure or tension) contribute to these surface forces?

Magnetic force and stress – 5



- Take the *surface* perpendicular to the x axis on the right-hand side of the box:

$$\boldsymbol{n} = \boldsymbol{e}_x : \qquad \boldsymbol{f}_{\mathrm{right}} = \boldsymbol{e}_x \cdot \mathbf{M},$$

$$f_{\mathrm{right},\,x} = -\frac{1}{4\pi} B_x B_z + \frac{1}{8\pi} B^2 = \frac{1}{8\pi} B^2, \quad f_{\mathrm{right},\,y} = f_{\mathrm{right},\,z} = 0.$$

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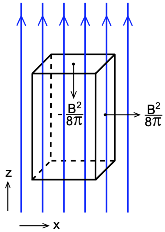

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- The stress exerted by the magnetic field at the top of the surface element is

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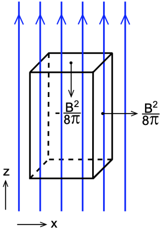

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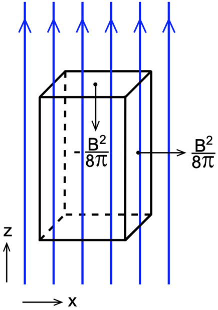
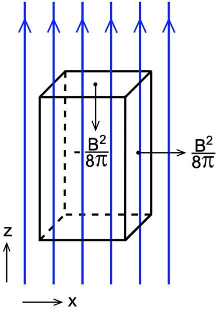
The stress is also perpendicular to the surface and of equal magnitude to that of the magnetic pressure exerted at the vertical surfaces, but of opposite sign!

Magnetic force and stress – 6



Conclusions:

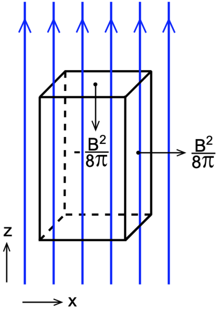
- The magnetic pressure causes the fluid volume to expand in the perpendicular directions to the magnetic field (in x and y for a field in z direction)



Conclusions:

- The magnetic pressure causes the fluid volume to expand in the perpendicular directions to the magnetic field (in x and y for a field in z direction)
- Because there are no magnetic monopoles (i.e., $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$), magnetic field lines have no ‘ends’. Hence, the contraction along the magnetic field under magnetic stress does not happen in practice because the tension at its top and bottom surfaces is exactly balanced by the tension in the magnetic lines continuing above and below the box.

Magnetic force and stress – 6



Conclusions:

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- The effects of tension in a magnetic field manifest themselves through the curvature of field lines.

Magneto-hydrodynamics

- For a collisional fluid on scales larger than the particle mean-free path and on time scales longer than the inverse plasma frequency, $\tau > \omega_{\mathrm{pl}}^{-1}$, the evolution of the magnetic vector field $\boldsymbol{B}$ is given by magneto-hydrodynamics (MHD).
- Ideal MHD assumes an inviscid (i.e., no viscosity), ideally conducting fluid.

AIP

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- Ideal MHD assumes an inviscid (i.e., no viscosity), ideally conducting fluid.
- To derive MHD, we add the Lorentz force to the momentum evolution equation (the Euler equation) and supplement the system of conservation equations of mass, momentum and entropy by the equation for magnetic induction, Eq. (2) without the diffusion term and obtain the equations of ideal MHD:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = 0,$$

$$\rho \left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} \right) = -\boldsymbol{\nabla} P + \boldsymbol{j} \times \boldsymbol{B} = -\boldsymbol{\nabla} \cdot \left[\left(P + \frac{\boldsymbol{B}^2}{8\pi} \right) \bar{\mathbf{1}} - \frac{1}{4\pi} \boldsymbol{B} \boldsymbol{B}^{\mathrm{T}} \right],$$

$$\frac{\partial s}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} s = 0,$$

$$\frac{\partial \boldsymbol{B}}{\partial t} - \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B}) = \mathbf{0}, \quad \text{subject to the constraint} \quad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0,$$

where $\rho = \rho(\boldsymbol{x}, t)$, $P = P(\boldsymbol{x}, t)$, $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x}, t)$, $\boldsymbol{j} = \boldsymbol{j}(\boldsymbol{x}, t)$, $s = s(\boldsymbol{x}, t)$, and $\boldsymbol{B} = \boldsymbol{B}(\boldsymbol{x}, t)$ are the density, pressure, velocity, electric current, entropy, and magnetic field.

Magnetic flux freezing – 1

- To show that the magnetic flux is "frozen" into the plasma, we start with the induction equation (2) without the diffusion term:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B}),$$

AIP

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- Using $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$, we obtain
$$\frac{\partial \boldsymbol{B}}{\partial t} = (\boldsymbol{B} \cdot \boldsymbol{\nabla})\boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{B} - (\boldsymbol{\nabla} \cdot \boldsymbol{v})\boldsymbol{B},$$
  which can be rearranged to yield
$$\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t} \equiv \frac{\partial \boldsymbol{B}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{B} = (\boldsymbol{B} \cdot \boldsymbol{\nabla})\boldsymbol{v} - (\boldsymbol{\nabla} \cdot \boldsymbol{v})\boldsymbol{B}.$$

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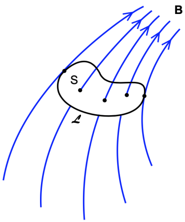
- Multiplying this equation by ρ^{-1} and rearranging terms yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{B}}{\rho}\right) = \frac{1}{\rho}\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t} - \frac{\boldsymbol{B}}{\rho^2}\frac{\mathrm{d}\rho}{\mathrm{d}t} = \left(\frac{\boldsymbol{B}}{\rho} \cdot \boldsymbol{\nabla}\right)\boldsymbol{v},$$

  This is the flux-freezing equation of magnetic fields.

Magnetic flux freezing – 2



Spruit (2013)

Flux freezing condition: $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\boldsymbol{B}}{\rho}\right) = \left(\dfrac{\boldsymbol{B}}{\rho}\cdot\boldsymbol{\nabla}\right)\boldsymbol{v}$

- Consider the evolution of $\delta\boldsymbol{x}$ which connects two neighboring points in the fluid:

$$\begin{aligned}
\Delta\boldsymbol{x}(t) &= \delta\boldsymbol{x} \\
\Delta\boldsymbol{x}(t+\Delta t) &= \delta\boldsymbol{x} + (\delta\boldsymbol{x}\cdot\boldsymbol{\nabla})\,\boldsymbol{v}\,\Delta t + \mathcal{O}(\Delta t^2) \\
\frac{\mathrm{d}\delta\boldsymbol{x}}{\mathrm{d}t} &= \frac{\Delta\boldsymbol{x}(t+\Delta t) - \Delta\boldsymbol{x}(t)}{\Delta t} = (\delta\boldsymbol{x}\cdot\boldsymbol{\nabla})\,\boldsymbol{v}
\end{aligned}$$

- $\boldsymbol{B}/\rho$ and $\delta\boldsymbol{x}$ satisfy the same *ordinary differential equation*, hence if initially $\delta\boldsymbol{x} = \varepsilon\boldsymbol{B}/\rho$, the same relation will hold for all times. If $\delta\boldsymbol{x}$ connects two particles on the same field line then they remain on the same field line.

Magnetic flux freezing – 3

Flux freezing condition: $$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{B}}{\rho}\right) = \left(\frac{\boldsymbol{B}}{\rho}\cdot\boldsymbol{\nabla}\right)\boldsymbol{v}$$

- What does this flux-freezing condition imply for a uniform contraction/expansion of the plasma?

AIP

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- What does this flux-freezing condition imply for a uniform contraction/expansion of the plasma?
- The plasma resides in a sphere of radius r and conserves mass and magnetic flux $\mathrm{d}\Phi = \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{A}$ (where $\mathrm{d}\boldsymbol{A}$ is the surface element on the sphere). Thus, both ρr^3 and Br^2 are constant and we obtain

$$B \equiv \sqrt{\langle \boldsymbol{B} \rangle} \propto r^{-2} \propto \rho^{\alpha_B}, \quad \alpha_B = \frac{2}{3},$$

for isotropic contraction or expansion, independent of the magnetic topology.

AIP

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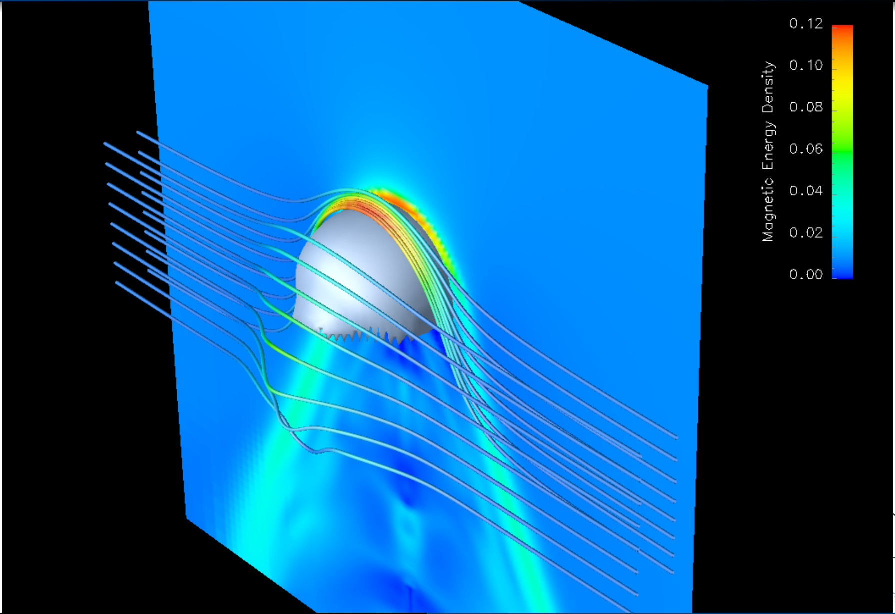
  for isotropic contraction or expansion, independent of the magnetic topology.
- Note that the scaling exponent α_B depends on the type of symmetry invoked during collapse (whether it is isotropic or not) and can differ for contraction along a homogeneous magnetic field ($\alpha_B = 0$) or perpendicular to it ($\alpha_B = 1$).
- Thus, flux freezing alone predicts a tight relation between B and ρ. Moreover, it has a surprising property called magnetic draping.

AIP

What is magnetic draping?

Interaction of an obstacle (Earth, star, galaxy, . . .) with a magnetized plasma



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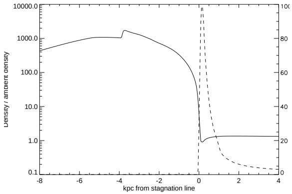
- Is magnetic draping similar to ram pressure compression?
  $\rightarrow$ no, the density is not increased in magnetic draping as shown by ideal MHD simulations *(right)*



Dursi & CP (2008)

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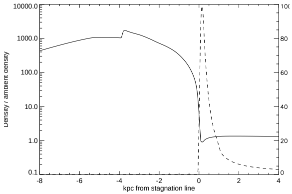
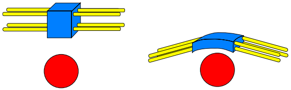
- Is magnetic flux still frozen into the plasma?
  yes, but plasma can also move along field lines while field lines get stuck at obstacle





Dursi & CP (2008)

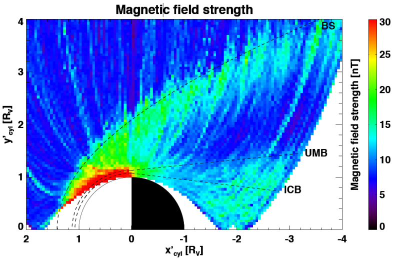
Applications of magnetic draping

- Solar-wind magnetic field is draped around the magnetopause of Earth: this protects Earth from cosmic rays during times of spin flip of the magnetic poles

- draping of solar-wind magnetic field at other moons and planets of the solar system: plasma physics

- hydrodynamic stability of underdense radio bubbles

- sharpness (T_e, n_e) of cold fronts: without B, smoothed out by diffusion and heat conduction on $\gtrsim 10^8$ yrs



Guicking et al. (2010): magnetic draping around Venus

- magnetic draping on spiral galaxies in galaxy clusters: method for detecting the orientation of cluster magnetic fields

Hydrodynamic waves

- Going back to our derivation of the dispersion relation for sound waves by perturbing the mass, momentum and entropy equation of a hydrodynamic fluid without conduction and viscosity. How many equations do you have and how many eigenvalues does the linearized system of equations allow for?

AIP

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- The hydrodynamic system of five equations reads (without viscosity and heat conduction)

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) &= 0, \\ \frac{\partial}{\partial t}(\rho \boldsymbol{v}) + \boldsymbol{\nabla} \cdot \left(\rho \boldsymbol{v}\boldsymbol{v}^{\mathrm{T}} + P\bar{\mathbf{1}}\right) &= \mathbf{0}, \\ \frac{\partial s}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} s &= 0. \end{aligned}$$

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- Combining the equations for mass and momentum yields (exercise sheet 3)

$$\omega^2 = \frac{\delta \hat{P}}{\delta \hat{\rho}} k^2 \qquad \Longrightarrow \qquad \omega = \pm\sqrt{\frac{\delta \hat{P}}{\delta \hat{\rho}}}\, k,$$

i.e., the sound wave is a degenerate solution and accounts for four eigenvalues.

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- Perturbing the entropy equation yields to first order in Fourier space

$$\mathrm{i}\omega \delta \hat{s} - \delta \hat{\boldsymbol{v}} \cdot \boldsymbol{\nabla} s_0 = 0$$
$$\Longrightarrow \quad \omega = 0 \quad \text{and} \quad s_0 = \text{const}.$$

The entropy mode is a compressible zero-frequency mode with eigenfunctions $\delta P = \delta \boldsymbol{v} = \delta \boldsymbol{B} = 0$ and $\delta T/T = -\delta\rho/\rho = 2\delta s/5$.

AIP

Magneto-hydrodynamic waves – 1

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AIP

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$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) &= 0, \\
\rho \left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} \right) = -\boldsymbol{\nabla} P + \boldsymbol{j} \times \boldsymbol{B} &= -\boldsymbol{\nabla} \cdot \left[\left(P + \frac{\boldsymbol{B}^2}{8\pi} \right) \bar{\mathbf{1}} + \frac{1}{4\pi} \boldsymbol{B}\boldsymbol{B}^{\mathrm{T}} \right], \\
\frac{\partial s}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} s &= 0, \\
\frac{\partial \boldsymbol{B}}{\partial t} - \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B}) &= \mathbf{0}, \quad \text{subject to the constraint} \quad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0,
\end{aligned}
$$

  where $\rho = \rho(\boldsymbol{x})$, $P = P(\boldsymbol{x})$, $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x})$, $\boldsymbol{j} = \boldsymbol{j}(\boldsymbol{x})$, $s = s(\boldsymbol{x})$, and $\boldsymbol{B} = \boldsymbol{B}(\boldsymbol{x})$ are the density, pressure, velocity, electric current, entropy, and magnetic field.

- There are a total of 8 equations: 5 hydrodynamics equations plus 3 components of the induction equation. However, the constraint equation, $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$, reduces the dimensionality to seven degrees of freedom.

AIP

Magneto-hydrodynamic waves – 2

In a magnetized plasma, there are seven different wave modes:

- 2 shear Alfvén waves: ***incompressible ($\delta\rho = 0$) and transverse polarized waves;*** restoring force provided by magnetic tension; propagate obliquely and parallel to $\boldsymbol{B}$ with $v_\mathrm{a} = B/\sqrt{4\pi\,\rho}$
- 2 fast magnetosonic waves: ***compressible, longitudinal waves;*** restoring force provided by (thermal and magnetic) pressure; propagate parallel and perpendicular to $\boldsymbol{B}$; equivalent to sound waves in high-β plasmas, where $\beta = P_\mathrm{th}/P_B = 2c_\mathrm{s}/v_\mathrm{a}$
- 2 slow magnetosonic waves: ***compressible, longitudinal waves;*** restoring force provided by thermal pressure; propagate only parallel to $\boldsymbol{B}$; equivalent to compressible Alfvén waves in high-β plasma
- entropy mode: zero-frequency mode with fluctuations in n and T such that the thermal pressure $P = \mathrm{const.}$

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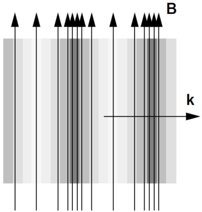
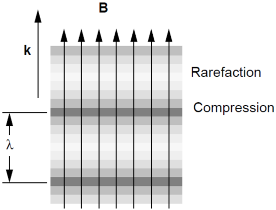
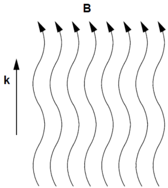


fast magnetosonic wave



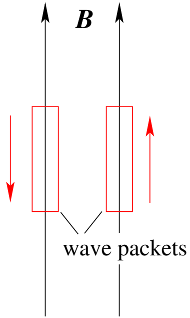
fast or slow sound wave



shear Alfvén wave

Mignone (2017)

Alfvénic turbulence – the picture



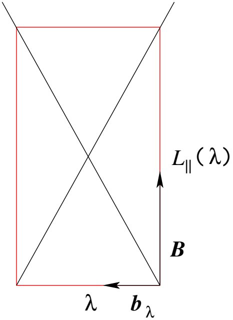
Interacting Alfvén wave packets.

- Alfvénic turbulence is incompressible:

$$\frac{\delta v_{\mathrm{A}}}{v_{\mathrm{A}}} = \frac{\delta B}{B}$$

- What happens when the two wave packets are interacting?
- The down-going packet causes field line wandering such that the upward going packet is broken apart after a distance $L_{||}(\lambda)$.
- In other words, the travel time across this wave package in the direction of the mean magnetic field equals the eddy turn-over time in the perpendicular direction.
- This gives rise to the critical balance condition of Alfvénic turbulence
  (Goldreich & Shridhar 95, 97, Lithwick & Goldreich 01)

Alfvénic turbulence – the scaling



Geometrical interpretation of the "critical balance" condition.

- The critical balance condition reads:

$$L_{\|} = \frac{\lambda B}{b_\lambda}$$

- In Kolmogorov turbulence, the energy flux of the fluctuating field at scale λ is constant, $b_\lambda^2 / t_\lambda = \text{const.}$ Equating the wave travel time along $\boldsymbol{B}$, $t_{\|}$, with the eddy turn-over time in the perpendicular direction, t_λ, we get

$$t_{\|} = \frac{L_{\|}}{v_a} = \frac{\lambda\, B}{v_a\, b_\lambda} = t_\lambda \propto b_\lambda^2,$$

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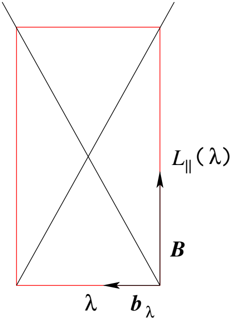

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- Because $B \propto v_\mathrm{a} = \mathrm{const.}$ in incompressible turbulence, we obtain the scaling of Alfvénic turbulence:

$$b_\lambda \propto \lambda^{1/3} \quad \text{or} \quad L_{\|} \propto \lambda^{2/3}\, L_\mathrm{MHD}^{1/3}$$

$\Rightarrow$ the smaller the scale λ, the more anisotropic is the turbulent scaling and the more elongated are the eddies ($L_{\|}/\lambda \propto \lambda^{-1/3}$) whose long axis is aligned with the local $\langle B \rangle$!

- **Non-thermal processes:**
  * radio relics and halos prove the existence of volume-filling magnetic fields and relativistic electrons in the ICM
  * the radial extent radio relics that propagate on the sky enables to estimate the magnetic field strength via a cooling length argument
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- **Magnetic waves and turbulence:**
  * MHD supports 4 modes: Alfvén waves, slow- and fast magnetosonic waves, and the zero-frequency entropy mode
  * MHD turbulence has an anisotropic cascade where eddies become more elongated towards smaller scales

AIP