Exercises for
The Physics of Galaxy Clusters

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Exercise sheet 2

To be uploaded to Moodle. Remember to put your name on the document. You may work in groups of up to 2 but every student should hand in his/her own solution sheet and indicate clearly who contributed to it. The exercises are based on the lecture notes. Thus, **studying the lecture notes carefully** will help you immensely in solving the exercises!

1. **Simplified Form of the Mass Function** (10 points)
   Masses and length scales are related by the mean background density
   $$M = \frac{4\pi}{3}\bar{\rho}R^3. \tag{1}$$
   The mass M_* corresponding to the scale R_* on which the variance becomes unity
   $$\sigma_*^2 = 4\pi \int_0^{k_*} \frac{k^2 \mathrm{d}k}{(2\pi)^3}\, P_\delta(k) = 1 \tag{2}$$
   is called the *nonlinear mass*.
   (a) Show that, for a power-law power spectrum with index n, $P_\delta(k) = Ak^n$, the variance can be written in the form
   $$\sigma^2 = \left(\frac{M_*}{M}\right)^{1+n/3}. \tag{3}$$
   (b) Discuss the special cases $n = 1$ and $n = -3$ and compare this to your qualitative drawing of σ^2 as a function of M by interpolating the asymptotic cases of the linear power spectrum.
   (c) Use this result to bring the Press-Schechter mass function into the form
   $$f(M,a)\mathrm{d}M \equiv \frac{\partial n_{\mathrm{PS}}(M,a)}{\partial M}\mathrm{d}M = \sqrt{\frac{2}{\pi}}\frac{\alpha\bar{\rho}\delta_{\mathrm{c}}}{M_* D_+} m^{\alpha-2} \exp\left(-\frac{\delta_{\mathrm{c}}^2}{2D_+^2}\, m^{2\alpha}\right)\mathrm{d}m, \tag{4}$$
   where $m = M/M_*$ and $\alpha = 1/2 + n/6$.

2. **The Nonlinear Mass** (10 points)
   (a) Using *Planck* data, the latest cosmic microwave background measurements yield $\sigma_8 \approx 0.8$ (i.e., σ on the scale of $8\,h^{-1}$ Mpc). Assume that $n = -1$ and estimate the nonlinear mass today.
   (b) How does the nonlinear mass evolve with time?
   (c) Calculate the present-day abundance of objects with mass M_* according to the Press-Schechter mass function (use $\Omega_{\mathrm{m},0} = 0.3$). Assuming, for simplicity, that these halos are randomly distributed through space, estimate the mean separation between these objects. Compare your estimate with the actual distance of the Milky Way to the Virgo cluster.

3. **Binding Energy** (10 points)

Consider a dark-matter halo with NFW density profile, i.e.,

$$\rho(r) = \frac{\rho_\mathrm{s}}{x(1+x)^2} \quad \text{with} \quad x = \frac{r}{r_\mathrm{s}}. \tag{5}$$

(a) Using physical reasoning, argue why the potential energy of the halo must be of the form

$$E_\mathrm{pot} = -\alpha \frac{G M_\mathrm{s}^2}{r_\mathrm{s}} \quad \text{with} \quad M_\mathrm{s} = 4\pi r_\mathrm{s}^3 \rho_\mathrm{s}, \tag{6}$$

where $\alpha > 0$ is a dimensionless constant.

(b) Confirm that the gravitational potential of an NFW halo is

$$\Phi(r) = -\frac{G M_\mathrm{s}}{r_\mathrm{s}} \frac{\ln(1+x)}{x}. \tag{7}$$

(c) Determine α by integrating to infinity for simplicity.