Exercises for
The Physics of Galaxy Clusters

Lecturer: Christoph Pfrommer

Exercise sheet 3

To be uploaded to Moodle. Remember to put your name on the document. You may work in groups of up to 2 but every student should hand in his/her own solution sheet and indicate clearly who contributed to it. The exercises are based on the lecture notes. Thus, **studying the lecture notes carefully** will help you immensely in solving the exercises!

1. **Sound Waves** (10 points)

   (a) By performing a perturbation analysis of the mass and momentum conservation equations, derive the dispersion relation for sound waves,

   $$\omega^2 = \frac{\delta \hat{P}}{\delta \hat{\rho}} k^2, \tag{1}$$

   where the hat indicates Fourier components.

   (b) Using this dispersion relation, derive the phase and group speed of sound waves.

   (c) Compare and discuss the different properties of sound and gravity waves.

2. **Gravity Waves.** (10 points)

   Let the total gas pressure and density be related by the isothermal sound speed, $P = c_{\mathrm{iso},0}^2\,\rho$ and let's assume a fixed gravitational field of the form

   $$\vec{g} = -\frac{g_0}{1 + z/z_0}\vec{e}_z. \tag{2}$$

   (a) Assuming hydrostatic equilibrium, derive the density stratification, i.e., $\rho = \rho(z, h)$ where $h = c_{\mathrm{iso},0}^2/g_0$ is the pressure scale height.

   (b) Take the limit $z_0 \to \infty$ and compute $\rho(z, h)$.

   (c) Now, compute the Brunt-Väisälä frequency for both atmospheres (finite z_0 and $z_0 \to \infty$). In which of the two atmospheres do you get g-mode trapping and at which height are they trapped?

   (d) Compute the Brunt-Väisälä frequency in the central cluster regions (which have a cuspy NFW density profile) and in the Earth's atmosphere to order of magnitude. To this end, you may take the limit $z \ll z_0$.

3. **Generalized Rankine-Hugoniot Shock Jumps** (10 points)

   Show, that a Galilean transformation of the Rankine-Hugoniot shock jump conditions from the shock to the laboratory rest system leads to the generalized Rankine-Hugoniot conditions of mass, momentum, and energy conservation at a shock,

   $$\begin{aligned} v_{\mathrm{s}}[\rho] &= [\rho u], \\ v_{\mathrm{s}}[\rho u] &= [\rho u^2 + P], \\ v_{\mathrm{s}}\left[\rho\frac{u^2}{2} + \varepsilon\right] &= \left[\left(\rho\frac{u^2}{2} + \varepsilon + P\right)u\right]. \end{aligned} \tag{3}$$

Here v_s and u denote the shock and the mean gas velocity measured in the laboratory rest system, $\varepsilon = \epsilon\rho$ is the thermal energy density, and we introduced the abbreviation $[F] = F_i - F_j$ for the jump of some quantity F across the shock.