Exercises for
The Physics of Galaxy Clusters

Lecturer: Christoph Pfrommer

Exercise sheet 4

To be uploaded to Moodle. Remember to put your name on the document. You may work in groups of up to 2 but every student should hand in his/her own solution sheet and indicate clearly who contributed to it. The exercises are based on the lecture notes. Thus, **studying the lecture notes carefully** will help you immensely in solving the exercises!

1. **Turbulent Scaling Laws** (10 points)

   This problem completes the *turbulence* topic we started two weeks ago. Consider the Navier-Stokes equation in the following compact form

   $$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\frac{\boldsymbol{\nabla} P}{\rho} + \nu \boldsymbol{\nabla}^2 \boldsymbol{v}, \tag{1}$$

   where $\mathrm{d}/\mathrm{d}t = \partial/\partial t + \boldsymbol{v} \cdot \boldsymbol{\nabla}$ is the Lagrangian time derivative and $\nu = \eta/\rho$ is the kinematic viscosity.

   (a) By introducing characteristic length L_0, velocity V_0, and density ρ_0 scales, rewrite the Navier-Stokes equation into dimensionless form. *Hint:* you also have to introduce a dimensionless time and a dimensionless Nabla operator using these three characteristic scales.

   You will find, that the dimensionless equation involves one number, the Reynolds number $\mathrm{Re} \equiv L_0 V_0/\nu$, that characterizes the flow and determines the structure of the solutions to this equation.

   (b) In the lectures, we introduced an energy flow rate per unit mass, $\dot{\epsilon} = v_\lambda^3/\lambda$, that is valid on all scales λ and constant (because energy does not accumulate at any intermediate scale). Hence, $\dot{\epsilon} = V^3/L$ has also the meaning of an energy injection rate into the turbulent cascade at the outer scale L. Defining the *Kolmogorov length* ℓ, show that this defines corresponding velocity and time scales,

   $$\ell \equiv \left(\frac{\nu^3}{\dot{\epsilon}}\right)^{1/4}, \quad v_\ell = (\dot{\epsilon}\nu)^{1/4}, \quad \tau_\ell = \left(\frac{\nu}{\dot{\epsilon}}\right)^{1/2}. \tag{2}$$

   What value has the Reynolds number Re at the *Kolmogorov length* ℓ and why?

   Work out the scaling of the following ratios, L_0/ℓ, V_0/v_ℓ, τ/τ_ℓ, and ϵ_0/ϵ_ℓ with the Reynolds number. We speak about a turbulent flow if the Reynolds number at the outer scale is $\mathrm{Re}(L) \gtrsim 10^3$. Interpret your ratios in the light of this requirement.

   (c) Do we have turbulent flows in the hot ($k_\mathrm{B}T = 10$ keV) intracluster medium ($n = 10^{-3}\,\mathrm{cm}^{-3}$), if the outer scale is $L_0 = 300$ kpc and we consider it to be a purely hydrodynamical system? Argue qualitatively, what you would expect to change, if we did add magnetic fields to the system.

2. **Cooling versus heating** (10 points)

   In this problem, you will calculate the X-ray luminosity of a cluster in which the gas density obeys a β profile.

   (a) Verify the expression for the X-ray luminosity of a cluster (Eqns. 3.189 and 3.190) with a beta profile

   $$n(r) = n_0 \left[1 + \left(\frac{r}{r_\mathrm{c}}\right)^2\right]^{-3\beta/2}, \tag{3}$$

   where $\beta = 2/3$ and 1.

   (b) Why does the X-ray luminosity depend on the core radius and not the virial radius? Why does it depend on the value of the density in the center and not the average cluster density? To answer both questions, derive a relation of how $\mathrm{d}L_\mathrm{X}/\mathrm{d}\ln r$ scales with radius r in the limiting regimes for small and large r.

   (c) How does the mass profile of such a cluster scale with radius at large radii for these two values of β? Compare this to the mass profiles of the singular isothermal sphere and the NFW profile.

   (d) Recall our remarks about the applicability of the mass profiles at large radii of order the virial radius and beyond and argue what happens at these radii so that the total mass of such a cluster is regularized.

3. **Order of magnitude energetics** (10 points)

   Please calculate to order of magnitude the energy (in erg) and energy deposition rate (in erg/s) for the following events:

   - bouncing tennis ball,
   - car crash,
   - explosion of atomic bomb,
   - stellar wind over the life time of a massive O-type star assuming a constant mass-loss rate $\dot{M} \sim 10^{-6}\,\mathrm{M}_\odot\,\mathrm{yr}^{-1}$,
   - type Ia supernova explosion until the end of the Sedov-Taylor phase ($t_\mathrm{ST} \sim 1000$ yr),
   - galaxy cluster merger, which dissipates gravitational energy over the dynamical time $t_\mathrm{dyn} \approx (G\bar{\rho})^{-1/2}$.

   Where necessary, assume suitable values for the corresponding masses, velocities, radii, and time scales.