Exercises for
The Physics of Galaxy Clusters

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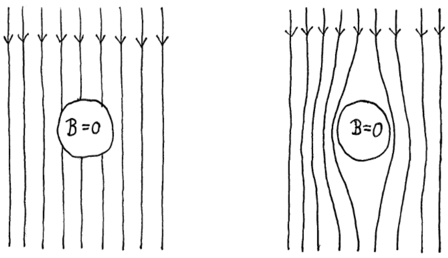
Exercise sheet 5

To be uploaded to Moodle. Remember to put your name on the document. You may work in groups of up to 2 but every student should hand in his/her own solution sheet and indicate clearly who contributed to it. The exercises are based on the lecture notes. Thus, **studying the lecture notes carefully** will help you immensely in solving the exercises!

1. **Magnetic compressibility** (10 points)

   (a) The following two drawings show possible configurations of magnetic field lines near a field-free inclusion (circular contour). Which of the two configurations is physical and why? Why is the other one not physical?



   (b) Show that under a uniform expansion, a uniform magnetic field stays uniform. To do so, use the following form of the flux-freezing equation (Eq. 3.289 in the lecture notes):

$$\frac{\partial \boldsymbol{B}}{\partial t} = (\boldsymbol{B} \cdot \boldsymbol{\nabla})\, \boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{\nabla})\, \boldsymbol{B} - (\boldsymbol{\nabla} \cdot \boldsymbol{v})\, \boldsymbol{B}$$

   and inspect the individual terms on the right hand side.

   (c) Now consider a uniform and isotropic expansion with $\boldsymbol{v} = \omega_{\mathrm{exp}} \boldsymbol{r}$, where ω_{exp} is the constant expansion frequency and $\boldsymbol{r}$ is the position vector from the origin of the coordinate system. Show that 1.) the magnetic field strength changes at a constant rate, $\partial_t \boldsymbol{B} = -2\, \omega_{\mathrm{exp}}\, \boldsymbol{B}$, and 2.) that $P_{\mathrm{mag}} = B^2/(8\pi) \propto \rho^{4/3}$.

2. **The Field length of an iso-cooling galaxy cluster** (20 points)

   For an optically thin, *ideal* gas that only cools by radiation, the rate of cooling energy density is given by

$$\dot{\varepsilon}_{\mathrm{th}} = n^2\, \Lambda(n, T),$$

   where $\varepsilon_{\mathrm{th}}$ is the thermal energy density of the gas, n is the particle number density, T is the gas temperature, and $\Lambda(n, T)$ is the volumetric cooling function (in units of $\mathrm{erg\, cm^3\, s^{-1}}$).

   (a) Assume a stratified atmosphere where the cooling time $t_{\mathrm{cool}} = \varepsilon_{\mathrm{th}}/\dot{\varepsilon}_{\mathrm{th}}$ *is constant* at every height z, and $n \equiv n(z)$ and $T \equiv T(z)$. Show that the corresponding differential temperature profile reads as:

$$\frac{\mathrm{d} \ln T}{\mathrm{d} \ln z} = \frac{\mathrm{d} \ln n}{\mathrm{d} \ln z} \left[\frac{1 + \Lambda_n}{1 - \Lambda_T} \right],$$

where $\Lambda_T = \partial \ln \Lambda \,/\, \partial \ln T$ and $\Lambda_n = \partial \ln \Lambda \,/\, \partial \ln n$, respectively.

(b) Additionally assume that the stratification fulfills the hydrostatic equilibrium condition:

$$\frac{\partial P_{\mathrm{th}}}{\partial z} = -g(z)\,\rho$$

where $g(z)$ describes the gravity and ρ is the gas mass density. Show that the temperature profile of such an atmosphere reads as:

$$\frac{\mathrm{d}T}{\mathrm{d}z} = -\frac{\mu m_{\mathrm{p}}}{k_{\mathrm{B}}} g(z) \left[1 + \frac{1-\Lambda_T}{1+\Lambda_n}\right]^{-1},$$

where μ is the mean molecular mass of the gas, m_{p} is the proton mass, and k_{B} is Boltzmann's constant.

(c) In galaxy clusters, the predominant radiative cooling mechanism is cooling by bremsstrahlung, which is well approximated by a cooling function of the form $\Lambda(T) = \Lambda_0\sqrt{T}$, where Λ_0 is a constant cooling efficiency. Show that:

$$\frac{\mathrm{d}T}{\mathrm{d}z} = -\frac{2}{3}\frac{\mu m_{\mathrm{p}}}{k_{\mathrm{B}}} g(z)$$

(d) Integrate the temperature profile assuming a gravity profile of the form

$$g(z) = g_0 \frac{z/a}{\sqrt{1+(z/a)^2}},$$

where $g_0 = k_{\mathrm{B}} T_0/(\mu m_{\mathrm{p}} H)$ is a constant of gravity, T_0 is a reference temperature in the cluster center, H is a scale height, and a is a gravitational smoothing length.

(e) For the resulting temperature profile, the corresponding density profile scales as

$$n = n_0 \left(\frac{T}{T_0}\right)^{1/2},$$

where n_0 is a reference number density in the cluster center. Show that the *Field length*, λ_{F}, in such an atmosphere scales linearly with temperature T.