Exercises for
The Physics of Galaxy Clusters

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Exercise sheet 6

To be uploaded to Moodle. Remember to put your name on the document. You may work in groups of up to 2 but every student should hand in his/her own solution sheet and indicate clearly who contributed to it. The exercises are based on the lecture notes. Thus, **studying the lecture notes carefully** will help you immensely in solving the exercises!

1. **Order of magnitude cosmic rays** (10 points)

   The relative cosmic ray (CR) pressure is defined as the ratio of CR pressure to thermal pressure, $X_{\rm CR} = P_{\rm CR}/P_{\rm th}$. The typical energy for a CR particle is of order 1 GeV and the magnetic field is of order 1 μG. Calculate to order of magnitude the number density ratio of thermal particles and CRs, $n_{\rm th}/n_{\rm CR}$ (i.e., how many thermal particles are there for one CR particle), and the ratio of CR-to-thermal Larmor radii, $L_{\rm CR}/L_{\rm th}$, for the following environments:

   (a) A Milky Way-like galaxy with $X_{\rm CR} \sim 1$ and a thermal energy per particle of $k_{\rm B}T \sim$ 1 eV.

   (b) A typical galaxy cluster with $X_{\rm CR} \sim 10^{-2}$ and a thermal energy per particle of $k_{\rm B}T \sim$ 10 keV.

2. **Adiabatic cosmic rays** (10 points)

   Introducing the dimensionless momentum $p = P_{\rm p}/(m_{\rm p}\, c)$, we assume that the differential cosmic ray (CR) particle momentum spectrum per volume element can be approximated by a single momentum power law above the minimum momentum q:

   $$f(p) = \frac{\mathrm{d}^2 N}{\mathrm{d}p\,\mathrm{d}V} = C\,p^{-\alpha}\,\theta(p-q),$$

   where $\theta(x)$ denotes the Heaviside step function. The CR density is then given by

   $$n_{\rm CR} = \int_0^\infty f(p)\,\mathrm{d}p.$$

   (a) Using Liouville's theorem, work out how the low-momentum cutoff q and CR normalization C changes upon an adiabatic density change from ρ_0 to ρ.

   (b) Imagine that CRs are accelerated at a strong cosmological structure formation shock with a relative CR pressure of $X_{\rm CR} = P_{\rm CR}/P_{\rm th} = 0.1$. Calculate $X_{\rm CR}$ in the ultra-relativistic limit after the composite of CRs and thermal gas has experienced adiabatic density increase by a factor of 10^3 from the warm-hot intergalactic medium to the cluster center.

   (c) (*Bonus*) Calculate the CR adiabatic index $\gamma_{\rm CR} = \mathrm{d}\ln P_{\rm CR}/\mathrm{d}\ln\rho$ and take the non-relativistic limit ($q \ll 1$ and $\alpha > 3$) and the ultra-relativistic limit ($q \to \infty$) of $\gamma_{\rm CR}$. To this end, use the definition of the CR pressure:

   $$P_{\rm CR} = \frac{m_{\rm p}c^2}{3}\int_0^\infty \mathrm{d}p\, f(p)\,\beta\, p = \frac{C\, m_{\rm p}c^2}{6}\,\mathcal{B}_{\frac{1}{1+q^2}}\left(\frac{\alpha-2}{2}, \frac{3-\alpha}{2}\right),$$

   where $\beta = v/c = p/\sqrt{1+p^2}$ is the dimensionless velocity of the CR particle and $\mathcal{B}_x(a,b)$ denotes the incomplete beta function, and $\alpha > 2$ is assumed.

3. **Dynamical friction of eccentric orbits** (10 points)

In the lecture you learnt about the effect of dynamic friction on circular orbits. In this task, you will investigate the effect of dynamic friction on eccentric orbits.

The eccentricity e of an orbit is defined as $e = (a - b)/(a + b)$, where a and b are the apocenter and pericenter of the orbit, respectively. These values are the roots for r of

$$\frac{1}{r^2} + \frac{2\,[\Phi(r) - E]}{L^2} = 0, \tag{1}$$

where Φ is the gravitational potential and E and L are the specific kinetic energy and specific angular momentum of the orbit, respectively.

(a) Assuming a singular isothermal sphere with the potential $\Phi(r) = v_\mathrm{c}^2 \ln(r/r_0)$, show that the maximum angular momentum $L_\mathrm{c}(E)$ for a given energy E reads as

$$L_\mathrm{c}(E) = v_\mathrm{c}\, r_0 \exp\left(\frac{E}{v_\mathrm{c}^2} - \frac{1}{2}\right) = v_\mathrm{c}\, r_\mathrm{c}(E), \tag{2}$$

with v_c the velocity and r_c the radius of a circular orbit.
Hint: Start from equation (1) and take the derivative with respect to r.

(b) The orbital circularity η of an eccentric orbit is defined as $\eta = L/L_\mathrm{c}(E(t))$. Dynamical friction transfers energy and angular momentum from the orbiting object to the background gas, causing η to change. Show that the temporal evolution of η can be expressed as

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = \eta \left[\frac{1}{L}\frac{\mathrm{d}L}{\mathrm{d}t} - \frac{1}{v_\mathrm{c}^2}\frac{\mathrm{d}E}{\mathrm{d}t}\right].$$

Hint: Use equation (2) to find an expression for the occurring term $\partial L_\mathrm{c}/\partial E$.

(c) Substituting $\mathrm{d}E/\mathrm{d}t = v\,\mathrm{d}v/\mathrm{d}t$ and $\mathrm{d}L/\mathrm{d}t = L/v\,\mathrm{d}v/\mathrm{d}t$ results in the following temporal evolution equation of the eccentricity:

$$\frac{\mathrm{d}e}{\mathrm{d}t} = \frac{\eta}{v}\frac{\mathrm{d}e}{\mathrm{d}\eta}\left[1 - \left(\frac{v}{v_\mathrm{c}}\right)^2\right]\frac{\mathrm{d}v}{\mathrm{d}t}.$$

Analyze this equation and describe how the different terms behave at apocenter and pericenter, respectively. From this, estimate how the eccentricity e of the orbit evolves over time.
Hint: $\mathrm{d}e/\mathrm{d}\eta < 0$ applies always.