Exercises for Cosmology (WS2013/14)

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Exercise sheet 1

Due: Oct. 22, 2013 16:00

To be handed in before the exercise class or emailed as a pdf or scanned hand-written document (via e-mail to walther@mpia.de). Remember to put your names on the document; groups of ≤ 3 are allowed.

1. **The Metric of Robertson-Walker space** (10 points)
   In the lecture, we defined the radial function

$$f_K(w) = \begin{cases} K^{-1/2}\sin(K^{1/2}w) & (K>0), \\ w & (K=0), \\ |K|^{-1/2}\sinh(|K|^{1/2}w) & (K<0), \end{cases} \tag{1}$$

   where K is a constant parametrizing the curvature of spatial hypersurfaces.

   (a) Show that the spatial line element of the Friedmann metric,

$$\mathrm{d}l^2 = \mathrm{d}w^2 + f_K^2(w)\mathrm{d}\Omega^2 \tag{2}$$

   can equivalently be written in the form

$$\mathrm{d}l^2 = \frac{\mathrm{d}r^2}{1-Kr^2} + r^2\mathrm{d}\Omega^2 \tag{3}$$

   (b) For both forms of the metric, calculate the surface area of a sphere with constant unit radius $w = 1$ resp. $r = 1$.

2. **Newtonian Friedmann Equations** (10 points)
   Consider a spherical sub-volume of radius $r(t)$, within an (infinite) expanding, homogeneous mass density distribution $\rho(t)$ of a idealized pressureless fluid. The radius $r(t) \equiv a(t) \cdot R$ (where $R = \text{const.}$) is defined to enclose constant mass as a function of time. At $t = t_0$, we have $r(t_0) = R$ and $\rho(t_0) = \rho_0$ and some expansion rate $\dot{a}(t_0)$.

   (a) What is the equation of motion for $a(t)$?

   (b) What is the total energy of the system E_{tot}, and what is the solution to this equation of motion for $E_{\text{tot}} = 0$?

   (c) If $a(t = 0) = 0$, how does the time t_0 since that 'big bang' (i.e. $a = 0$) compare to $1/\dot{a}(t_0)$? Does that t_0 depend on the choice of R?

(to be continued on the back)

3. **Loitering Universes** (10 points)
Consider a Friedmann model with non-relativistic matter ($\rho_\mathrm{m} \propto a^{-3}$) and a cosmological constant ($\rho_\Lambda \propto a^0$) with densities relative to critical of Ω_m and Ω_Λ, respectively.

   (a) By considering the 'second Friedmann equation' for $\ddot{a}$ show that there is an inflexion point in the scale factor evolution when
$$(1+z)^3 = \frac{2\Omega_{\Lambda,0}}{\Omega_{\mathrm{m},0}}. \tag{4}$$

   (b) Show that the age of the Universe becomes infinite when $\dot{a} \to 0$ at the inflexion point or
$$27\,\Omega_{\mathrm{m},0}^2\Omega_{\Lambda,0} = 4(\Omega_{\mathrm{m},0} + \Omega_{\Lambda,0} - 1)^3. \tag{5}$$
Models near this one, with a long period where a is nearly constant, are known as loitering models. The extreme loitering model is Einstein's static universe, also known as Eddington-Lemaitre universes, and are the reason Einstein introduced Λ in the first place. Models with long loitering phases are heavily disfavored by the classical tests (and eventually by Olber's paradox!).

4. **Necessity of a Big Bang** (10 points)
Again, consider the universe of the previous problem (you may also use the results obtained there).

   (a) What region of the $\Omega_{\mathrm{m},0} - \Omega_{\Lambda,0}$ plane leads to models with no 'big bang' (i.e., a does not tend to zero at early times)?

   (b) Show that such universes have a maximum redshift z_max with
$$\Omega_{\mathrm{m},0} \leq \frac{2}{z_\mathrm{max}^2(z_\mathrm{max}+3)}. \tag{6}$$
Since we observe objects at $z \gtrsim 5$ and $\Omega_{\mathrm{m},0} \gg 0.01$ argue that such models are ruled out.