

Exercises for Cosmology (WS2013/14)

Lecturer: Christoph Pfrommer; Exercises: Michael Walther

Exercise sheet 4

Due: Nov. 12, 2013 16:00

To be handed in before the exercise class or emailed as a pdf or scanned hand-written document (via e-mail to walther@mpia.de). Remember to put your names on the document; groups of ≤ 3 are allowed.

1. Non-relativistic Gas (10 points)

- (a) **Decoupling.** Show that for a particle of mass m that decouples at temperature T_d with $mc^2 \gg kT_d$ the distribution remains thermal but with a modified temperature T and chemical potential μ

$$T = T_d \left(\frac{a_d}{a} \right)^2 \quad \text{and} \quad \mu = mc^2 + (\mu - mc^2) \frac{T}{T_d}. \quad (1)$$

Hint: employ the non-relativistic limit of the phase space distribution (don't forget the rest-mass contribution to the energy), recall that particle momenta scale as $p \propto a^{-1}$ and use the conservation of the comoving number density, $N = na^3 = \text{const.}$

- (b) **Adiabatic Index.** By analogy to the calculation in the script, show that the grand-canonical potential for a non-relativistic gas well after decoupling (see equation (1)) can be written in the form

$$\Phi = -AVT^{5/2}, \quad (2)$$

where $A > 0$ is a constant. Use this to show that pressure and entropy scale with temperature as

$$P = A(kT)^{5/2}, \quad (3)$$

$$S = \frac{5}{2} AVk(kT)^{3/2}. \quad (4)$$

Use the relations $U = TS - PV$ and $dU + PdV = 0$ to show that

$$P \propto V^{-5/3}, \quad (5)$$

i.e., the adiabatic index of a non-relativistic gas is $\gamma = 5/3$.

2. **QCD Phase Transition** (10 points)

In case there are multiple particle species in *thermal equilibrium*, we can define an effective degeneracy factor (or effective statistical weight) of

$$g_* \equiv \sum_{i=\text{bosons}} \tilde{g}_i + \frac{7}{8} \sum_{i=\text{fermions}} \tilde{g}_i = \sum_{i=\text{bosons}} g_i + \sum_{i=\text{fermions}} g_i. \quad (6)$$

We will focus on the quantum chromodynamics (QCD) phase transition, which takes place at $kT = 200$ MeV. You may wish to use the fact that below 200 MeV, the “light” particles are e ($g = 7/2$), γ ($g = 2$), ν ($g = 21/4$), μ ($g = 7/2$), and π ($g = 3$).

- (a) Justify the values of the corresponding statistical weights g_i .
- (b) Calculate the mass, in M_\odot , contained within the horizon at the QCD phase transition. You may approximate the horizon distance as ct and the background density at the critical density at that time.

3. **Cosmological Bounds on the Neutrino Mass** (10 points)

A neutrino with mass less than 1 MeV will decouple when $kT \simeq 1$ MeV at which temperature the effective statistical weight is given by $g_* = 10.75$. If the neutrino does not interact then n/s will be conserved. For a single massive species with $g = 2$ show that the bound $\Omega_\nu h^2 < 1$ gives

$$m_\nu c^2 < 92 \text{ eV}. \quad (7)$$