Exercises for
Cosmology (WS2014/15)

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Exercise sheet 1

Due: Oct. 24, 2014 11:00

To be handed in before the exercise class or emailed as a pdf or scanned hand-written document (via e-mail to walther@mpia.de). Remember to put your names on the document; you can work in groups of ≤ 2 but every student should hand in his/her own solution sheet and indicate clearly who contributed to it.

1. **Newtonian Friedmann Equations** (10 points)
   Consider a spherical sub-volume of radius $r(t)$, within an (infinite) expanding, homogeneous mass density distribution $\rho(t)$ of a idealized pressureless fluid. The radius $r(t) \equiv a(t) \cdot R$ (where $R = \text{const.}$) is defined to enclose constant mass as a function of time. At $t = t_0$, we have $r(t_0) = R$ and $\rho(t_0) = \rho_0$ and some expansion rate $\dot{a}(t_0)$.

   (a) Use the equation of motion for a test mass on the sphere's surface to derive an equation of motion for $a(t)$? How does this compare to the 2nd Friedmann equation?

   (b) What is the total energy of a test mass on the sphere's surface. Which cosmological parameter is represented by E_{tot}, and what is the solution to this equation of motion for $E_{\text{tot}} = 0$?

   (c) If $a(t = 0) = 0$, how does the time t_0 since that 'big bang' (i.e. $a = 0$) compare to $1/\dot{a}(t_0)$? Does that t_0 depend on the choice of R?

2. **Loitering Universes and the Big Bang** (10 points)
   Consider a Friedmann model with non-relativistic matter ($\rho_{\text{m}} \propto a^{-3}$) and a cosmological constant ($\rho_\Lambda \propto a^0$) with densities relative to critical of Ω_{m} and Ω_Λ, respectively.

   (a) By considering the 'second Friedmann equation' for $\ddot{a}$ show that there is an inflexion point in the scale factor evolution when

   $$(1+z)^3 = \frac{2\Omega_{\Lambda,0}}{\Omega_{\text{m},0}}. \tag{1}$$

   (b) Show that the age of the Universe becomes infinite when $\dot{a} \to 0$ at the inflexion point and

   $$27\,\Omega_{\text{m},0}^2\Omega_{\Lambda,0} = 4(\Omega_{\text{m},0} + \Omega_{\Lambda,0} - 1)^3. \tag{2}$$

   Models near this one, with a long period where a is nearly constant, are known as loitering models. The extreme loitering model is Einstein's static universe, also known as Eddington-Lemaitre universes, and are the reason Einstein introduced Λ in the first place. Models with long loitering phases are heavily disfavored by the classical tests (and eventually by Olber's paradox!).

(to be continued on the back)

(c) What region of the $\Omega_{\mathrm{m},0} - \Omega_{\Lambda,0}$ plane leads to models with no 'big bang' (i.e., a does not tend to zero at early times)?

(d) Show that such universes have a maximum redshift $z_{\max}$ with

$$\Omega_{\mathrm{m},0} \leq \frac{2}{z_{\max}^2 (z_{\max} + 3)}. \tag{3}$$

Since we observe objects at $z \gtrsim 5$ and $\Omega_{\mathrm{m},0} \gg 0.01$ argue that such models are ruled out. Thus, this proves the necessity of a 'big bang' if we can describe our Universe with such a Friedmann model.