Exercises for Cosmology (WS2014/15)

Lecturer: Christoph Pfrommer; Exercises: Michael Walther

Exercise sheet 10

Due: Jan. 9, 2014 11:00

To be handed in before the exercise class or emailed as a pdf or scanned hand-written document (via e-mail to walther@mpia.de). Remember to put your names on the document; you can work in groups of ≤ 2 but every student should hand in his/her own solution sheet and indicate clearly who contributed to it. All problems discuss relevant content for the final exam; however, you are only required to solve two out of the three problems and hence you can only earn a **maximum of 20 points per problem set** (which can be accumulated from any of the three problems).

1. **Halo Constraints on the Nature of Dark Matter** (10 points)

   The following provides an argument against the possibility that massive neutrinos constitute the majority of the dark matter. The velocity dispersion of stars in dwarf galaxies suggest that these objects may be dark matter dominated. The central parts of the Draco dwarf galaxy can be modeled as a cored isothermal sphere, with King radius $r_0 = 150$ pc and one-dimensional velocity dispersion $\sigma = 9\,\mathrm{km\ s^{-1}}$. For an isothermal sphere, r_0 and σ are related to the central density, ρ_0, by

   $$r_0 \equiv \left(\frac{9\sigma^2}{4\pi G\rho_0}\right)^{1/2}. \tag{1}$$

   If most of the mass of Draco resides in a single neutrino species and its anti-neutrino, find a lower limit on the neutrino mass in eV c^{-2} so that the Pauli exclusion principle is not violated. Show that this mass is inconsistent with the neutrino mass required for $\Omega_\nu \leq 1$ (problem 3 on exercise sheet 4).
   Hint: one possible way to go about this is to compare the microscopic and macroscopic phase space densities.

2. **Binding Energy** (10 points)

   Consider a dark-matter halo with NFW density profile, i.e.,

   $$\rho(r) = \frac{\rho_\mathrm{s}}{x(1+x)^2} \quad \text{with} \quad x = \frac{r}{r_\mathrm{s}}. \tag{2}$$

   (a) Using physical reasoning, argue why the potential energy of the halo must be of the form

   $$E_\mathrm{pot} = -\alpha\frac{GM_\mathrm{s}^2}{r_\mathrm{s}} \quad \text{with} \quad M_\mathrm{s} = 4\pi r_\mathrm{s}^3\rho_\mathrm{s}, \tag{3}$$

   where $\alpha > 0$ is a dimensionless constant.

   (b) Confirm that the gravitational potential of an NFW halo is

   $$\Phi(r) = -\frac{GM_\mathrm{s}}{r_\mathrm{s}}\frac{\ln(1+x)}{x}. \tag{4}$$

   (c) Determine α by integrating to infinity for simplicity.

3. **Gas in an NFW Halo** (10 points)

The NFW density profile diverges in the center, i.e., for $x \to 0$. Gas filled in the halo's gravitational potential Φ satisfies Euler's equation

$$\frac{\nabla p_{\rm gas}}{\rho_{\rm gas}} = -\nabla \Phi(r), \tag{5}$$

where $p_{\rm gas}$ is the gas pressure.

(a) Assuming an isothermal and ideal gas, show that the gas density profile is

$$\rho_{\rm gas} = A \exp\left(-\frac{\bar{m}\Phi}{kT}\right), \tag{6}$$

where T is the temperature, k is Boltzmann's constant, $\bar{m}$ is the mean particle mass, and A is a constant.

(b) Using Equation (4), show that

$$-\frac{\bar{m}\Phi}{kT} = 3\frac{\ln(1+x)}{x} \tag{7}$$

if the gas is in equilibrium with the gravitational potential.

(c) Is the gas density finite in the halo's center? Compare the density profiles of gas and dark matter and explain the differences.

(d) What happens to the gas-to-dark matter mass density ratio $\rho_{\rm gas}/\rho$ at large radii? Is this a realistic behavior and if not, what would have to be changed in the model to make it more realistic?