Exercises for Cosmology (WS2014/15)

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Exercise sheet 11

Due: Jan. 16, 2015 11:00

To be handed in before the exercise class or emailed as a pdf or scanned hand-written document (via e-mail to walther@mpia.de). Remember to put your names on the document; you can work in groups of ≤ 2 but every student should hand in his/her own solution sheet and indicate clearly who contributed to it. All problems discuss relevant content for the final exam; however, you are only required to solve two out of the three problems and hence you can only earn a **maximum of 20 points per problem set** (which can be accumulated from any of the three problems).

1. **Cosmological Parameter Estimation from the CMB** (10 points)

   In this problem, we will derive an estimate of the angular scale on which we expect to find the first acoustic peak in the CMB fluctuation spectrum, using a simplified model of recombination. We will then explore which cosmological parameter this scale depends most sensitively on. For the purposes of the problem, we will assume that recombination happens instantaneously at $z = 1100$, and hence that this redshift also corresponds to the surface of last scattering. Furthermore, we will assume that prior to recombination, matter and radiation are strongly coupled and act as a single fluid with sound-speed that we approximate as $c_\mathrm{s} = c/\sqrt{3}$. After recombination, we assume that there is no further coupling between matter and radiation.

   (a) The conformal time $\eta(t)$ is defined as

   $$\eta(t) = \int_0^t \frac{\mathrm{d}t'}{a(t')}. \tag{1}$$

   Analytically calculate the conformal time at which the CMB was formed. Why can you neglect curvature and cosmological constant terms?

   (b) The first acoustic peak in the CMB occurs on the scale of the "sound horizon", the maximum distance that sound waves in the photon-baryon fluid could have propagated before the CMB was formed. Use your result of part (a) to estimate the size of the *comoving* sound horizon in units of h^{-1} Mpc.

   (c) Analytically calculate the angular size (in degrees) corresponding to this peak, as observed at the present day for a critical universe with $(\Omega_{\mathrm{m},0}, \Omega_{\Lambda,0}) = (1.0, 0)$ and for an open Universe with $(\Omega_{\mathrm{m},0}, \Omega_{\Lambda,0}) = (0.3, 0)$.

   (d) Now suppose that we live in a low-density, flat universe with a cosmological constant, $(\Omega_{\mathrm{m},0}, \Omega_{\Lambda,0}) = (0.3, 0.7)$. Numerically calculate the angular size on which we expect to find the first acoustic peak and compare your answer to part (c). To which cosmological parameter is the peak position most sensitive? Explain the leading effect by using a suitable drawing.

(e) A great tool for getting more insight into cosmological parameter dependencies of the CMB fluctuations are the tutorials on Wayne Hu's web page:
`http://background.uchicago.edu/index.html`
In particular the *Cosmological Parameter Animations* are well suited to understand how the CMB power spectrum changes upon varying baryon and matter content, curvature, dark energy etc. Pick your favorite parameter (other than the one we studied explicitly in this problem) and explain qualitatively how and why the CMB power spectrum changes upon varying that parameter.

2. **Optical Depth due to Reionization** (10 points)

After recombination the Universe eventually became *reionized*, which allows CMB photons to once again scatter off electrons. The Thomson scattering optical depth from an observer at $z = 0$ out to a redshift of z can be written as

$$\tau(z) = \int_{D_{\rm prop}(0)}^{D_{\rm prop}(z)} \sigma_{\rm T} n_{\rm e} {\rm d}D_{\rm prop} = \int_0^z \sigma_{\rm T} n_{\rm e}(z') \frac{c}{(1+z')H(z')}\,{\rm d}z', \tag{2}$$

where $\sigma_{\rm T}$ is the Thomson scattering cross-section and $n_{\rm e}(z')$ is the physical electron number density at a redshift $z = z'$. One effect of this scattering is to suppress the size of any CMB fluctuations by mixing together photons from cold and warm regions. To a first approximation, the amplitudes of the perturbations are multiplied by a factor $e^{-\tau}$, with τ given by the formula above.

Assume that the Universe reionizes instantly at a redshift $z_{\rm reion}$ and that the intergalactic medium has a primordial composition (mass fractions of $X = 0.76$ for hydrogen and $1 - X = 0.24$ for helium). Compute $e^{-\tau}$ as a function of $z_{\rm reion}$ using the canonical cosmological model of $(\Omega_{\rm b,0}, \Omega_{\rm m,0}, \Omega_{\Lambda,0}, h) = (0.045, 0.3, 0.7, 0.7)$. You may use a suitable approximation in your final integral. What does the fact that we see *any* CMB fluctuations imply regarding the range of possible values for $z_{\rm reion}$?

3. **Inflation with a Quadratic Potential** (10 points)

In the slow-roll approximation, the evolution equations governing our inflaton, ϕ, can be written as

$$H^2 = \frac{8\pi G}{3} V(\phi), \qquad 3H\dot{\phi} = -\frac{{\rm d}V}{{\rm d}\phi}, \tag{3}$$

where $V(\phi)$ is the potential. Consider a model for inflation where

$$V(\phi) = \frac{1}{2} m^2 \phi^2. \tag{4}$$

(a) Solve the evolution equations for this case, and determine the time evolution of the scale factor a with the initial conditions $a = a_{\rm i}$ and $\phi = \phi_{\rm i}$ at $t = 0$.

(b) For what range of ϕ values is the solution inflationary?

(c) What condition must be obeyed by $\phi_{\rm i}$ to ensure that an expansion of at least 10^{30} takes place?