Exercises for
Cosmology (WS2014/15)

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Exercise sheet 2

Due: Oct. 31, 2014 11:00

To be handed in before the exercise class or emailed as a pdf or scanned hand-written document (via e-mail to walther@mpia.de). Remember to put your names on the document; you can work in groups of ≤ 2 but every student should hand in his/her own solution sheet and indicate clearly who contributed to it. All problems discuss relevant content for the final exam; however, you are only required to solve two out of the three problems and hence you can only earn a **maximum of 20 points per problem set** (which can be accumulated from any of the three problems).

1. **Cosmological Epochs** (10 points)
   We have seen in the lecture that during the course of the evolution of the universe different components have dominated the expansion history. Here we assume a canonical cosmological model of $(\Omega_{\mathrm{m},0}, \Omega_{\Lambda,0}, h) = (0.3, 0.7, 0.7)$.

   (a) **Matter-Radiation Equality.** The energy density of the CMB can be written in terms of the CMB temperature, T_{CMB}, as

   $$u_{\mathrm{CMB}} = \frac{\pi^2}{15} \frac{(kT_{\mathrm{CMB}})^4}{(\hbar c)^3}. \tag{1}$$

   With the help of this equation, write down an expression for $\Omega_{\mathrm{r}}(a)$ in terms of T_{CMB}. (You can neglect the contribution from neutrinos that are subdominant.) If $T_{\mathrm{CMB}} = 2.726\,\mathrm{K}$ at $z = 0$, what is the value of $\Omega_{\mathrm{r},0}$? Calculate the redshift z_{eq} at which the energy densities of matter and radiation are equal.

   (b) **Dominance of the Cosmological Constant.** Calculate the redshift $z_{\mathrm{eq},\Lambda}$ at which the energy densities of matter and the cosmological constant are equal.

   (c) **Accelerated Expansion.** Rewrite the 'second Friedmann equation' for $\ddot{a}$ and show that this defines a deceleration parameter

   $$q(t) \equiv \frac{a\ddot{a}}{\dot{a}^2} = -\frac{\Omega_{\mathrm{m}}(a)}{2} + \Omega_{\Lambda}(a), \tag{2}$$

   which is valid for times well after the radiation-dominated epoch. Use this equation to determine the redshift at which the universe started its accelerated expansion phase. Compare and discuss your result in comparison to $z_{\mathrm{eq},\Lambda}$.

(to be continued on the back)

2. **Standard Candles and Cosmological Parameters** (10 points)
In 2011, the Nobel Prize in Physics was awarded to Saul Perlmutter, Brian P. Schmidt and Adam G. Riess "for the discovery of the accelerating expansion of the Universe through observations of distant supernovae" of type Ia. Here, we will show why their measurement is a very sensitive probe of the expansion history and enabled the discovery of the accelerated expansion.

   (a) Imagine, we have a large number of well calibrated supernovae at $z = 0.5$ which fix the luminosity distance $D_L(z = 0.5)$ to high accuracy. Such a measurement constrains us to a "degeneracy line" in the $\Omega_{\mathrm{m},0} - \Omega_{\Lambda,0}$ plane. What is the slope of this line at the point $\Omega_{\mathrm{m},0} = 0.3 = 1 - \Omega_{\Lambda,0}$?
   Hint: by making use of $D_L(z = 0.5) = \text{const.}$, you may want to find a relation between $\Omega_{\mathrm{m},0}$ and $\Omega_{\Lambda,0}$ that you should vary while allowing for *non-zero* curvature. Only after variation, you should evaluate the degeneracy direction around the fiducial flat model with $\Omega_{\mathrm{m},0} + \Omega_{\Lambda,0} = 1$ (otherwise there would be no degeneracy in the $\Omega_{\mathrm{m},0} - \Omega_{\Lambda,0}$ plane)! Hence, you may want to start by Taylor expanding $f_K(w)$ around $\Omega_K = 0$.

   (b) How does this degeneracy direction compare to that obtained from cluster surveys (which are sensitive to the amount of matter in the universe, i.e., $\Omega_{\mathrm{m},0} \simeq \text{const.}$) and from studies of the anisotropies in the cosmological microwave background (which measure the characteristic angular scale of a known physical scale and redshift, i.e., they are sensitive to the curvature, $1 - \Omega_K = \Omega_{\mathrm{m},0} + \Omega_{\Lambda,0}$)? What is the benefit of combining those different surveys?

3. **Angular Diameter Distance** (10 points)

   (a) Show that the angular diameter distance from $z = 0$ to $z > 0$ reaches a *maximum* for
$$\dot{a} = \frac{c f_K'[w(a)]}{f_K[w(a)]}. \tag{3}$$

   (b) For an Einstein-de Sitter universe, show that this maximum is reached at a redshift of $z_{\max} = 5/4$. What are the implications of such a maximum and why does it occur?

   (c) Imagine we take a cluster of galaxies of size 1 Mpc and place it at successively higher redshifts in the cosmological concordance model of $(\Omega_{\mathrm{m},0}, \Omega_{\Lambda,0}, h) = (0.3, 0.7, 0.7)$. At which redshift does the cluster attain its minimum angular size, and what is that size? You may solve the equation numerically.