Exercises for Cosmology (WS2014/15)

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Exercise sheet 3

Due: Nov. 7, 2014 11:00

To be handed in before the exercise class or emailed as a pdf or scanned hand-written document (via e-mail to walther@mpia.de). Remember to put your names on the document; you can work in groups of ≤ 2 but every student should hand in his/her own solution sheet and indicate clearly who contributed to it. All problems discuss relevant content for the final exam; however, you are only required to solve two out of the three problems and hence you can only earn a **maximum of 20 points per problem set** (which can be accumulated from any of the three problems).

1. **Alcock-Paczynski Test** (10 points)
   Imagine we take a sphere of radius 1 Mpc and place it at $z = 1$. The sphere subtends an angle $\Delta\theta$ and a redshift width Δz.

   (a) What is the ratio $\Delta\theta/\Delta z$ in the cosmological concordance model of $(\Omega_{\mathrm{m},0}, \Omega_{\Lambda,0}, h) = (0.3, 0.7, 0.7)$.

   (b) What is the ratio, if we set $\Omega_{\Lambda,0} = 0$ (and keep $\Omega_{\mathrm{m},0} = 0.3$)? The sensitivity of this ratio to Λ is the basis of a sequence of cosmological tests to constrain the cosmological constant known collectively as the "Alcock-Paczynski test".

2. **Dark Energy Equation of State** (10 points)
   Suppose dark energy has an equation of state $P = w\rho c^2$, where we now allow $w(z)$ to be a function of redshift (for a cosmological constant, $w = -1$). Show that the Hubble expansion rate (well after the radiation-dominated epoch) is now given by

$$\frac{H^2(z)}{H_0^2} = \Omega_{\mathrm{m},0}(1+z)^3 + \Omega_{\mathrm{X},0} \exp\left[3\int_0^z [1 + w(x)]\, \mathrm{d}\ln(1+x)\right]. \tag{1}$$

   Here, we assume $(\Omega_{\mathrm{m},0}, \Omega_{\mathrm{X},0}, h) = (0.3, 0.7, 0.7)$, where $\Omega_{\mathrm{X},0}$ is the fraction of critical density contributed by dark energy today. Many experiments in the next decade aim at constraining $w(z)$ to infer clues on the nature of dark energy and employ various cosmological tests. To understand how sensitive these measurements have to be, plot (numerically) the

   (a) age of the universe

   (b) luminosity distance

   (c) comoving volume

   (d) $\Delta z/\Delta\theta$ (for an object of size 1 Mpc, the Alcock-Paczynski test of problem 1)

   as a function of redshift for 4 different models: $w = -1, w = -1/3, w = -0.5 + 0.1z, w = -0.5 - 0.05z$, for $0 < z < 5$.

One problem is that these measures involve integrals of $w(z)$ over redshift, and thus there may be degeneracies between different models of $w(z)$. Show if we had a very precise clock which could measure age differences between objects at different redshifts (passively evolving stellar populations?), $w(z)$ in principle could be directly determined from differential ages:

$$H_0^{-2}\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = \frac{[H_0^{-1}(\mathrm{d}z/\mathrm{d}t)]^2}{(1+z)}\left[\frac{5}{2}+\frac{3}{2}w(z)\right] - \frac{3}{2}\Omega_{\mathrm{m},0}(1+z)^4 w(z). \tag{2}$$

3. **Baryon-to-photon ratio** (10 points)
Consider a gas composed of particles obeying Bose-Einstein statistics that is in thermal equilibrium at a temperature T. As we discussed in the lecture, the particle number density for such a gas can be written as

$$n = \frac{g}{(2\pi\hbar)^3}\int_0^\infty \frac{4\pi p^2 \mathrm{d}p}{\exp\left[E(p)/kT\right]-1}, \tag{3}$$

where g is the statistical weight for an individual particle (containing contributions from e.g. the spin degeneracy factor), p is the momentum of the particle and $E(p)$ is the energy of the particle, which can be written in terms of the momentum as

$$E(p) = \sqrt{p^2c^2 + m^2c^4}, \tag{4}$$

where m is the particle mass.

(a) Show that for a photon gas,

$$n = \frac{2\zeta(3)}{\pi^2}\left(\frac{kT}{\hbar c}\right)^3, \tag{5}$$

where ζ is the Riemann zeta function.
Hint: you may find the following identity useful

$$\int_0^\infty \frac{x^m \mathrm{d}x}{e^x - 1} = m!\,\zeta(m+1). \tag{6}$$

(b) The measured value of the CMB temperature at the present day is $T_{\mathrm{CMB}} = 2.726$ K. Use this fact, together with Equation (5), to solve for the present number density of CMB photons. How does this value evolve with redshift?

(c) Compute the current mean number density of baryons in the Universe. Assume, for simplicity, that $\Omega_{\mathrm{b},0} = 0.05$ and $h = 0.7$.

(d) Use your answers to parts (b) and (c) to compute the current value of the baryon-to-photon ratio.

(e) Suppose that some process acting at high redshift creates additional photons, at the rate of one photon per baryon, and that thereafter these photons do not interact with the baryons. Would this create a significant distortion in the thermal spectrum of the CMB? Justify your answer.