Exercises for Cosmology (WS2014/15)

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Exercise sheet 4

Due: Nov. 14, 2014 11:00

To be handed in before the exercise class or emailed as a pdf or scanned hand-written document (via e-mail to walther@mpia.de). Remember to put your names on the document; you can work in groups of ≤ 2 but every student should hand in his/her own solution sheet and indicate clearly who contributed to it. All problems discuss relevant content for the final exam; however, you are only required to solve two out of the three problems and hence you can only earn a **maximum of 20 points per problem set** (which can be accumulated from any of the three problems).

1. **Non-relativistic Gas** (10 points)

   (a) **Decoupling.** Show that for a particle of mass m that decouples at temperature $T_{\rm d}$ with $mc^2 \gg kT_{\rm d}$ the distribution remains thermal but with a modified temperature T and chemical potential μ

   $$T = T_{\rm d}\left(\frac{a_{\rm d}}{a}\right)^2 \qquad \text{and} \qquad \mu = mc^2 + (\mu - mc^2)\,\frac{T}{T_{\rm d}}. \tag{1}$$

   Hint: employ the non-relativistic limit of the phase space distribution (don't forget the rest-mass contribution to the energy), recall that particle momenta scale as $p \propto a^{-1}$ and use the conservation of the comoving number density, $N = na^3 = \text{const}$.

   (b) **Adiabatic Index.** By analogy to the calculation in the script, show that the grand-canonical potential for a non-relativistic gas well after decoupling (see equation (1)) can be written in the form

   $$\Phi = -AVT^{5/2}, \tag{2}$$

   where $A > 0$ is a constant. Use this to show that pressure and entropy scale with temperature as

   $$P = A(kT)^{5/2}, \tag{3}$$
   $$S = \frac{5}{2}AVk(kT)^{3/2}. \tag{4}$$

   Use the relations $U = TS - PV$ and $\mathrm{d}U + P\mathrm{d}V = 0$ to show that

   $$P \propto V^{-5/3}, \tag{5}$$

   i.e., the adiabatic index of a non-relativistic gas is $\gamma = 5/3$.

2. **QCD Phase Transition** (10 points)
   In case there are multiple particle species in *thermal equilibrium*, we can define an effective degeneracy factor (or effective statistical weight) of

$$g_* \equiv \sum_{i=\text{bosons}} \tilde{g}_i + \frac{7}{8} \sum_{i=\text{fermions}} \tilde{g}_i = \sum_{i=\text{bosons}} g_i + \sum_{i=\text{fermions}} g_i. \tag{6}$$

   We will focus on the quantum chromodynamics (QCD) phase transition, which takes place at $kT = 200$ MeV. You may wish to use the fact that below 200 MeV, the "light" particles are e ($g = 7/2$), γ ($g = 2$), ν ($g = 21/4$), μ ($g = 7/2$), and π ($g = 3$).

   (a) Justify the values of the corresponding statistical weights g_i.

   (b) Calculate the mass, in $M_\odot$, contained within the horizon at the QCD phase transition. You may approximate the horizon distance as ct and the background density at the critical density at that time.

3. **Cosmological Bounds on the Neutrino Mass** (10 points)
   A neutrino with mass less than 1 MeV will decouple when $kT \simeq 1$ MeV at which temperature the effective statistical weight is given by $g_* = 10.75$. If the neutrino does not interact then n/s will be conserved. For a singe massive species with $g = 2$ show that the bound $\Omega_\nu h^2 < 1$ gives

$$m_\nu c^2 < 92 \text{ eV}. \tag{7}$$