Exercises for Cosmology (WS2014/15)

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Exercise sheet 6

Due: Nov. 28, 2014 11:00

To be handed in before the exercise class or emailed as a pdf or scanned hand-written document (via e-mail to walther@mpia.de). Remember to put your names on the document; you can work in groups of ≤ 2 but every student should hand in his/her own solution sheet and indicate clearly who contributed to it. All problems discuss relevant content for the final exam; however, you are only required to solve two out of the three problems and hence you can only earn a **maximum of 20 points per problem set** (which can be accumulated from any of the three problems).

1. **Growth of Structure** (10 points)

   We start with the linear perturbation equation for the density contrast of pressureless dark matter,

   $$\ddot{\delta} + 2H\dot{\delta} = 4\pi G\bar{\rho}_{\mathrm{m}}\delta, \tag{1}$$

   where $\bar{\rho}_{\mathrm{m}}$ is the mean background density, $H = H(a)$, and each dot denotes a time derivative.

   (a) Transforming the time derivatives to derivatives with respect to scale factor a, show that Equation (1) can be written as

   $$\left(a^3 H\delta'\right)' = \frac{3\Omega_{\mathrm{m},0}H_0^2}{2Ha^2}\delta, \tag{2}$$

   where the prime denotes the derivative with respect to a.

   (b) Show that $\delta_1 = E(a)$ is one solution of Equation (2), *provided* that $H^2 \equiv H_0^2 E^2(a)$ is of the form

   $$H^2 = \frac{A_0}{a^3} + \frac{A_1}{a^2} + A_2, \tag{3}$$

   where A_0, A_1, and A_2 are arbitrary constants. Argue why this form of H^2 is of importance for cosmology.

   (c) Use the *ansatz* $\delta_2 = Ef$ to show that δ_2 is the other solution of Equation (2), provided that

   $$f' = \frac{1}{a^3E^3}. \tag{4}$$

   Hint: underway, employ the fact that E is a solution of Equation (2), which is an example of the d'Alembert reduction. Thus

   $$\delta_2 = E(a)\int_0^a \frac{\mathrm{d}\tilde{a}}{\tilde{a}^3E^3(\tilde{a})} \tag{5}$$

   is the other solution of the linear growth equation.

2. **The Mészaros Effect** (10 points)

   Dark matter perturbations on scales much smaller than the horizon do not grow during the radiation-dominated epoch. In this problem, you will demonstrate why this is the case.

   (a) The growth of dark matter perturbations on scales much larger than the dark matter free-streaming scale and much smaller than the horizon scale is governed by Equation (1). Justify why the radiation energy density does not appear explicitly in this expression.

   (b) Write down an expression for H as a function of $\bar{\rho}_{\rm m}$ and the radiation energy density $\bar{\rho}_{\rm r}$ during the radiation-dominated epoch. Assume that we can neglect the effect of the baryons and that the curvature and Λ terms are negligible.

   (c) Use Equation (1) together with H during the radiation-dominated epoch to show that we can construct the following equation for δ,

$$\delta'' + \frac{2+3y}{2y(1+y)}\delta' - \frac{3}{2y(1+y)}\delta = 0, \tag{6}$$

   where the prime denotes the derivative with respect to a and $y \equiv \bar{\rho}_{\rm m}/\bar{\rho}_{\rm r}$.

   (d) Show that this equation has a solution of the form

$$\delta = \delta_0 \left(y + \frac{2}{3} \right). \tag{7}$$

   In the radiation-dominated epoch, $y \ll 1$, and so we see that δ barely increases during this epoch.

3. **Evolution of Potential and Velocity Perturbations** (10 points)

   (a) Starting from the Poisson equation and the equation for velocity perturbations,

$$\vec{\nabla}^2 \delta\Phi = 4\pi G \bar{\rho}_{\rm m} a^2 \delta, \qquad \delta\vec{v} = -\frac{2f(\Omega)}{3aH\Omega}\vec{\nabla}\delta\Phi, \tag{8}$$

   derive how potential and velocity perturbations evolve in an Einstein-de-Sitter universe.

   (b) Use Equations (8) to relate the power spectra of $\delta\Phi$ and $\delta\vec{v}$ to the power spectrum $P(k)$ of density fluctuations.
   Hint: this is most easily done in Fourier space.

   (c) Interpret these results. What do they imply for the typical scales found in potential, velocity, and density perturbations?