Exercises for
Cosmology (WS2014/15)

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Exercise sheet 8

Due: Dec. 12, 2015 11:00

To be handed in before the exercise class or emailed as a pdf or scanned hand-written document (via e-mail to walther@mpia.de). Remember to put your names on the document; you can work in groups of ≤ 2 but every student should hand in his/her own solution sheet and indicate clearly who contributed to it. All problems discuss relevant content for the final exam; however, you are only required to solve two out of the three problems and hence you can only earn a **maximum of 20 points per problem set** (which can be accumulated from any of the three problems).

1. **Spherical Collapse with Cosmological Constant** (10 points)

   We consider a spherical overdensity, which expands, slows down, turns around, and collapses in a spatially flat universe with a cosmological constant. We normalize the scale factor a and the radius of the overdensity R by their values at turn-around,

   $$x \equiv \frac{a}{a_{\mathrm{ta}}}, \qquad y \equiv \frac{R}{R_{\mathrm{ta}}}, \tag{1}$$

   introduce the dimensionless time, $\tau \equiv H_{\mathrm{ta}} t$ (where H_{ta} is the Hubble function at turn-around), and define ζ as the relative overdensity inside the sphere at turn-around.

   (a) Use both(!) Friedmann equations to show that the dimensionless scale factor x and the radius y obey the differential equations

   $$x' = \sqrt{\frac{\omega}{x} + (1-\omega)x^2}, \qquad y'' = -\frac{\omega\zeta}{2y^2} + (1-\omega)y, \tag{2}$$

   where ω is the matter-density parameter at turn-around, $\omega \equiv \Omega_{\mathrm{m}}(a_{\mathrm{ta}})$ and the prime denotes a derivative with respect to the dimensionless time τ.
   Hint: you may find it useful to follow the (simplified) derivation of such a pair of differential equations in the Einstein-de Sitter universe as it is being done in the revised form of the lecture notes by Bartelmann in Appendix A.

   (b) Use the boundary condition $y'(x=1) = 0$ to obtain the first integral of y''

   $$y' = \left[\zeta\omega\left(\frac{1}{y} - 1\right) + (1-\omega)\left(y^2 - 1\right)\right]^{1/2}, \tag{3}$$

   which needs to be solved numerically.

2. **Angular Momenta of Galaxies** (10 points)

   The angular momentum of a galaxy (L) can be expressed in terms of the dimensionless spin parameter

   $$\lambda \equiv \frac{L|E|^{1/2}}{GM^{5/2}}, \tag{4}$$

   where E is the binding energy and M is the mass of the galaxy. A system with appreciable rotational support has $\lambda \sim 1$. Observations suggest that disk galaxies have $\lambda \approx 0.4 - 0.5$.

   One possible way of understanding the angular momenta of galaxies is through the effects of tidal torques during gravitational collapse. Simulations indicate that this process can provide an initial λ of $\lambda_{\rm i} \approx 0.05$, which is only about 10% of the observed value. During the collapse of gas the binding energy increases due to cooling, while mass and angular momentum remain the same. This will allow λ to increase as $\lambda \propto |E|^{1/2}$ and (possibly) reach observed values.

   (a) Consider the collapse of a homogeneous overdense sphere of mass M and initial radius R_0 from its initial state that is expanding with the Hubble flow. Show that the collapse time is given by

   $$t_{\rm coll} \sim \pi \left(\frac{R_0^3}{2GM} \right)^{1/2}. \tag{5}$$

   (b) Examine the idea of the increase of the spin parameter λ during the collapse in the absence of dark matter halos and show that it is not viable by estimating the initial size that the gas cloud would need and the timescale for it to collapse to the scales associated with galaxies ($M_{\rm disk} \sim 10^{11}{\rm M}_\odot$ and $R_{\rm disk} \sim 10\,{\rm kpc}$).

   (c) Next, consider the same process in the presence of a dark matter halo. Show that the difficulties in part (b) can be circumvented and a sufficiently high value of λ can result if $M_{\rm disk} \approx 0.1 M_{\rm halo}$.

3. **Gravitational Lensing at Galaxies** (10 points)

   Suppose that a galaxy has an effective radius $r_{\rm g} \simeq 10\,h^{-1}{\rm kpc}$, and an effective cross section of $\pi r_{\rm g}^2$. Furthermore assume a constant comoving number density of galaxies $n_{\rm g} \simeq 0.02\,h^3{\rm Mpc}^{-3}$.

   (a) Show that the optical depth due to galaxies for an object at redshift z in a flat, matter dominated model is

   $$\tau \simeq 10^{-2} \left[(1+z)^{3/2} - 1 \right]. \tag{6}$$

   (b) Keep the model flat by lowering $\Omega_{\rm m}$ and introducing Λ. At what value of Ω_Λ does the optical depth become unity? Evaluate your answer analytically and express it as a function of redshift. This sensitivity to path length is the principle behind limits on Λ from e.g., gravitational lensing.