Exercises for
Cosmology & Galaxies (Summer Term 2018)

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Exercise sheet 2

Due: May 1, 2018, 16:00

To be emailed as a pdf or scanned hand-written document (via e-mail to kehlert@aip.de). Remember to put your names on the document; you can work in groups of ≤ 2 but every student should hand in his/her own solution sheet and indicate clearly who contributed.

Note that some solutions require the numerical application of integration and/or root-finding algorithms as well as plotting results. Experienced students may use their programming language of choice. For students with little or no previous knowledge, a small (python) tutorial covering the essential tools necessary for the exercises is available on Christoph Pfrommer's website.

1. **Angular Diameter Distance** (8 points)

   (a) Show that the angular diameter distance from $z = 0$ to $z > 0$ reaches a *maximum* for
   $$\dot{a} = \frac{c f_K'[w(a)]}{f_K[w(a)]}. \tag{1}$$

   (b) For an Einstein-de Sitter universe, show that this maximum is reached at a redshift of $z_{\max} = 5/4$. What are the implications of such a maximum and why does it occur?

   (c) Imagine we take a cluster of galaxies of size 1 Mpc and place it at successively higher redshifts in the cosmological concordance model of $(\Omega_{\mathrm{m},0}, \Omega_{\Lambda,0}, h) = (0.3, 0.7, 0.7)$. At which redshift does the cluster attain its minimum angular size, and what is that size? You may solve the equation numerically.

2. **Dark Energy Equation of State** (10 points)
   Suppose dark energy has an equation of state $P = w\rho c^2$, where we now allow $w(z)$ to be a function of redshift (for a cosmological constant, $w = -1$).Here, we assume $(\Omega_{\mathrm{m},0}, \Omega_{\mathrm{X},0}, h) = (0.3, 0.7, 0.7)$, where $\Omega_{\mathrm{X},0}$ is the fraction of critical density contributed by dark energy today.

   (a) Show that the Hubble expansion rate (well after the radiation-dominated epoch) is now given by
   $$\frac{H^2(z)}{H_0^2} = \Omega_{\mathrm{m},0}(1+z)^3 + \Omega_{\mathrm{X},0} \exp\left[3\int_0^z [1 + w(x)]\,\mathrm{d}\ln(1+x)\right]. \tag{2}$$

   Many experiments in the next decade aim at constraining $w(z)$ to infer clues on the nature of dark energy and employ various cosmological tests.

   To understand how sensitive these measurements have to be, plot (numerically) the following quantities as a function of redshift for 4 different models: $w = -1, w =$

$-1/3, w = -0.5 + 0.1z, w = -0.5 - 0.05z$, for $0 < z < 5$ (note, the integration algorithm in python only accepts infinity as an upper(!) integration boundary):

(b) age of the universe,

(c) luminosity distance,

(d) comoving volume.

(e) One problem is that these measures involve integrals of $w(z)$ over redshift, and thus there may be degeneracies between different models of $w(z)$. Show if we had a very precise clock which could measure age differences between objects at different redshifts (passively evolving stellar populations?), $w(z)$ in principle could be directly determined from differential ages:

$$H_0^{-2}\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = \frac{[H_0^{-1}(\mathrm{d}z/\mathrm{d}t)]^2}{(1+z)}\left[\frac{5}{2}+\frac{3}{2}w(z)\right] - \frac{3}{2}\Omega_{\mathrm{m},0}(1+z)^4 w(z). \quad (3)$$

3. **Decoupling of a non-relativistic Gas** (6 points)
Show that for a particle of mass m that decouples at temperature T_d with $mc^2 \gg kT_\mathrm{d}$ the distribution remains thermal but with a modified temperature T and chemical potential μ

$$T = T_\mathrm{d}\left(\frac{a_\mathrm{d}}{a}\right)^2 \qquad \text{and} \qquad \mu = mc^2 + (\mu - mc^2)\, cTT_\mathrm{d}. \quad (4)$$

Hint: employ the non-relativistic limit of the phase space distribution $f(p) = \{\exp[\epsilon(p) - \mu)/kT] \pm 1\}^{-1}$ (compare with Eq. 1.84 in the script for definitions and don't forget the rest-mass contribution to the energy ϵ), recall that particle momenta scale as $p \propto a^{-1}$ and use the conservation of the comoving number density, $N = na^3 = \text{const}$.

4. **Saha Equation** (6 points)
Here, we are going to calculate the redshift of the release of the CMB radiation. To express the (free) electron number density in terms of the baryon number density, we need an expression of the ionization degree x. According to Saha's equation it obeys

$$\frac{x^2}{1-x} = \frac{0.26}{\eta}\left(\frac{m_e c^2}{kT}\right)^{3/2} \mathrm{e}^{-\chi/kT}, \quad (5)$$

where $\eta = 6 \times 10^{-10}$ is the baryon-to-photon ratio of the Universe, m_e is the electron mass, c is the light speed, k is the Boltzmann constant, and χ is the ionization potential of a hydrogen atom.

Let us now estimate at which temperature T recombination sets in, $x = 0.95$, and at what temperature it is largely done, $x = 10^{-3}$.

(a) Argue why Saha's equation needs to be solved numerically and why it may be more appropriate to take the logarithm of this equation. Use a method of your choice to solve this equation and send your program code to Kristian by email (kehlert@mpia.de). E.g., you can use the Newton-Raphson algorithm, which finds the roots of the equation $f(x) = 0$. Starting from an educated guess x_0, roots are refined by the iteration

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}. \tag{6}$$

Using a sketch, show how this iteration works (or the method of your choice). Discuss possible cases where such an algorithm may fail to converge.

(b) What is the final resulting temperature for the two values of $x = 0.95$ and $x = 10^{-3}$? What are the corresponding redshifts? How much time did reionization take (between these two redshifts)?

(c) Discuss (just a few sentences) why we only 'see' the CMB photons now that were released when the matter had become almost completely neutral ($x \ll 1$).