Exercises for
Cosmology & Galaxies (Summer Term 2018)

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Exercise sheet 3

Due: May 22, 2018, 16:00

To be emailed as a pdf or scanned hand-written document (via e-mail to kehlert@aip.de). Remember to put your names on the document and number your pages; you can work in groups of ≤ 2 but every student should hand in his/her own solution sheet and indicate clearly who contributed.
A total of 40 points can be achieved when completing this sheet of which 30 points will count as the maximum attainable.

1. **Growth of Structure** (10 points)

We start with the linear perturbation equation for the density contrast of pressureless dark matter,

$$\ddot{\delta} + 2H\dot{\delta} = 4\pi G\bar{\rho}_{\mathrm{m}}\delta, \tag{1}$$

where $\bar{\rho}_{\mathrm{m}}$ is the mean background density, $H = H(a)$, and each dot denotes a time derivative.

(a) Transforming the time derivatives to derivatives with respect to scale factor a, show that Equation (1) can be written as

$$\left(a^3 H\delta'\right)' = \frac{3\Omega_{\mathrm{m},0}H_0^2}{2Ha^2}\delta, \tag{2}$$

where the prime denotes the derivative with respect to a.

(b) Show that $\delta_1 = E(a)$ is one solution of Equation (2), *provided* that $H^2 \equiv H_0^2 E^2(a)$ is of the form

$$H^2 = \frac{A_0}{a^3} + \frac{A_1}{a^2} + A_2, \tag{3}$$

where A_0, A_1, and A_2 are arbitrary constants. Argue why this form of H^2 is of importance for cosmology.

(c) Use the *ansatz* $\delta_2 = Ef$ to show that δ_2 is the other solution of Equation (2), provided that

$$f' = \frac{1}{a^3E^3}. \tag{4}$$

Hint: underway, employ the fact that E is a solution of Equation (2), which is an example of the d'Alembert reduction. Thus

$$\delta_2 = E(a)\int_0^a \frac{\mathrm{d}\tilde{a}}{\tilde{a}^3E^3(\tilde{a})} \tag{5}$$

is the other solution of the linear growth equation.

2. **The Mészaros Effect** (10 points)

Dark matter perturbations on scales much smaller than the horizon do not grow during the radiation-dominated epoch. In this problem, you will demonstrate why this is the case.

(a) The growth of dark matter perturbations on scales much larger than the dark matter free-streaming scale and much smaller than the horizon scale is governed by Equation (1). Justify why the radiation energy density does not appear explicitly in this expression.

(b) Write down an expression for H as a function of $\bar{\rho}_{\rm m}$ and the radiation energy density $\bar{\rho}_{\rm r}$ during the radiation-dominated epoch. Assume that we can neglect the effect of the baryons and that the curvature and Λ terms are negligible.

(c) Use Equation (1) together with H during the radiation-dominated epoch to show that we can construct the following equation for δ,

$$\delta'' + \frac{2+3y}{2y(1+y)}\delta' - \frac{3}{2y(1+y)}\delta = 0, \qquad (6)$$

where the prime denotes the derivative with respect to a and $y \equiv \bar{\rho}_{\rm m}/\bar{\rho}_{\rm r}$.

(d) Show that this equation has a solution of the form

$$\delta = \delta_0 \left(y + \frac{2}{3} \right). \qquad (7)$$

In the radiation-dominated epoch, $y \ll 1$, and so we see that δ barely increases during this epoch.

3. **The Mass Function** (10 points)

Masses and length scales are related by the mean background density

$$M = \frac{4\pi}{3}\bar{\rho}R^3. \qquad (8)$$

The mass M_* corresponding to the scale R_* on which the variance becomes unity

$$\sigma_*^2 = 4\pi \int_0^{k_*} \frac{k^2 {\rm d}k}{(2\pi)^3} P_\delta(k) = 1 \qquad (9)$$

is called the *nonlinear mass.*

(a) Show that, for a power-law power spectrum with index n, $P_\delta(k) = Ak^n$, the variance can be written in the form

$$\sigma^2 = \left(\frac{M_*}{M} \right)^{1+n/3}. \qquad (10)$$

(b) Discuss the special cases $n = 1$ and $n = -3$ and draw qualitatively σ^2 as a function of M by interpolating the asymptotic cases of the linear power spectrum.

(c) Use this result to bring the Press-Schechter mass function into the form

$$f(M,a)\mathrm{d}M \equiv \frac{\partial n_{\mathrm{PS}}(M,a)}{\partial M}\mathrm{d}M = \sqrt{\frac{2}{\pi}}\frac{\alpha\bar{\rho}\delta_{\mathrm{c}}}{M_* D_+} m^{\alpha-2}\exp\left(-\frac{\delta_{\mathrm{c}}^2}{2D_+^2}\,m^{2\alpha}\right)\mathrm{d}m, \tag{11}$$

where $m = M/M_*$ and $\alpha = 1/2 + n/6$.

4. **The Nonlinear Mass** (10 points)

(a) Using *Planck* data, the latest cosmic microwave background measurements yield $\sigma_8 \approx 0.8$ (i.e., σ on the scale of $8\,h^{-1}$ Mpc). Assume that $n = -1$ and estimate the nonlinear mass today.

(b) How does the nonlinear mass evolve with time?

(c) Calculate the present abundance of objects with mass M_* according to the Press-Schechter mass function (use $\Omega_{\mathrm{m},0} = 0.3$). Assuming, for simplicity, that these halos are randomly distributed through space, estimate the mean separation between these objects. Compare your estimate with the actual distance of the Milky Way to the Virgo cluster.