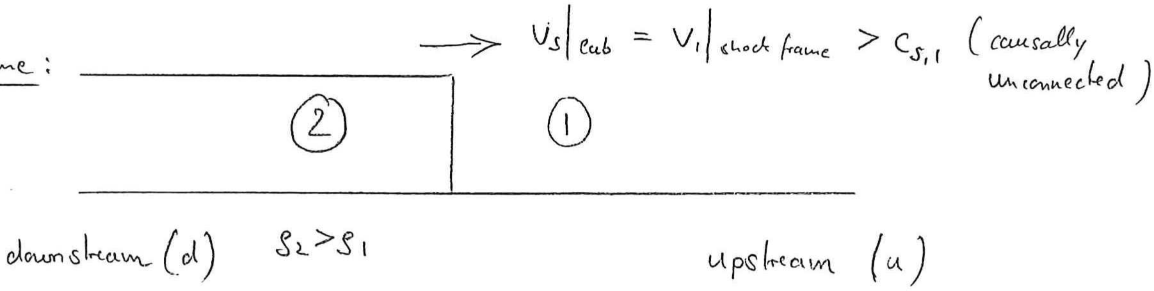
Cosmic Rays

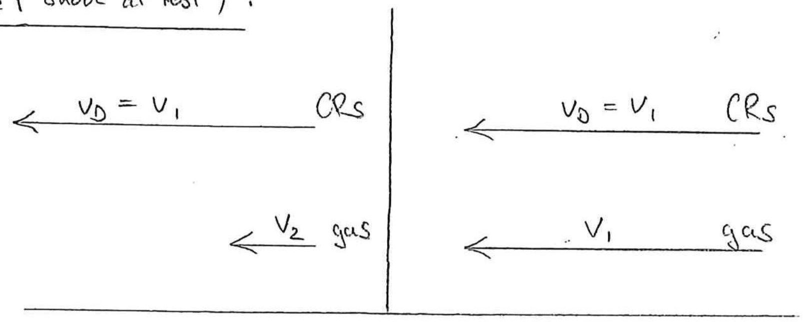
Diffusive Shock Acceleration (First-order Fermi Acceleration)

- at a collisionless shock, electrons and protons (ions) can be accelerated to highly relativistic energies; consider the geometry:

1. lab frame:

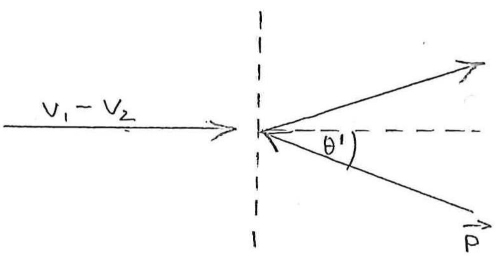


2. shock frame (shock at rest):



$\rho_1 v_1 = \rho_2 v_2$

3. CR rest frame in post-shock region: (boosting shock frame by v_1 to the right)



here, the postshock medium acts as an approaching mirror; scattering off this mirror (i.e. macroscopic magnetic irregularities or MHD waves) energizes the particle;

- we can work out a condition for a particle to scatter downstream and to reach upstream again, where it was initially advected with the flow; to this end, we transform from the lab to the post-shock frame and back to the lab frame; accretion shocks are non-relativistic (NR) shocks with $v_s < c$, while shocks in AGN jets are relativistic

- the Lorentz boost is taken along $c\vec{\beta} = \vec{v}_1 - \vec{v}_2$; we define the parallel component of the particle momentum in the lab frame $p_{\parallel} \equiv \cos\theta\, p \equiv \mu p$; in the post-shock frame, the cosmic ray energy and momentum are

$$\begin{pmatrix} E' \\ p' \end{pmatrix} = \begin{pmatrix} \gamma & \pm\beta\gamma \\ \pm\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} E \\ \mu p \end{pmatrix} = \begin{pmatrix} \gamma E \pm \beta\gamma\mu p \\ \pm\beta\gamma E + \gamma\mu p \end{pmatrix} \underset{\approx}{\overset{\substack{\text{NR} \\ \text{limit}}}{\downarrow}} \begin{pmatrix} E + \beta\mu p \\ \beta E + \mu p \end{pmatrix}$$

- in the last step, we assumed a non-relativistic shock and adopted the "+" sign for a Lorentz transformation in the direction of the cosmic ray's parallel momentum ; after colliding with the "magnetic mirror" p' is reversed to $-p'$ and E' remains unchanged ; transferring back to the lab frame, we find

$$\begin{pmatrix} E'' \\ p'' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} E' \\ -p' \end{pmatrix} \overset{\substack{\text{NR} \\ \text{limit}}}{\approx} \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} E' \\ -p' \end{pmatrix} = \begin{pmatrix} E + \beta\mu p + \beta(\beta E + \mu p) \\ -\beta(E + \beta\mu p) - \beta E - \mu p \end{pmatrix}$$

$$\overset{\substack{\text{NR} \\ \text{limit}}}{\approx} \begin{pmatrix} E + 2\beta\mu p \\ -2\beta E - \mu p \end{pmatrix} \equiv \begin{pmatrix} E + \delta E \\ p + \delta p \end{pmatrix}$$

- for a relativistically moving cosmic ray, we have a condition to be advected downstream, scatter there and reach the upstream again:

$$\delta E = \delta p c = 2(v_1 - v_2)\mu p \quad ; \quad \mu = \cos\theta > 0$$

hence the relative energy increase during an "udu" half cycle is

$$\frac{\delta p}{p} = \frac{2(v_1 - v_2)\mu}{c} > 0 \qquad (\mu > 0)$$

- energy is also increased during a "dud" half cycle :

$$\frac{\delta p}{p} = -\frac{2(v_1 - v_2)\mu}{c} > 0 \qquad (\mu < 0)$$

$\Rightarrow$ there is a net energy increase of a particle that finishes a full cycle "udud"

- to obtain the energy increase of a population of cosmic rays, we have to integrate this expression (3.161) for individual particles that are distributed according to a distribution function $f(\vec{x},\vec{p})$ over phase space; we conveniently employ spherical coordinates in momentum space $(p, \mu = \cos\theta, \varphi_p)$ and cylindrical coordinates in configuration space (r, φ, z) where the first 2 coordinates span an (arbitrary) cross section σ of the shock front and the z coordinate is transformed into a time coordinate via $dz = \vec{n}_s \cdot \vec{v}\, dt$ ($\vec{n}_s$ denotes the shock normal); hence, we have the differential volume $d^3x = d\sigma\, \vec{n}_s \cdot \vec{v}\, dt = d\sigma\, v\mu\, dt$; for a half cycle "udu", we get

$$\frac{\delta p}{p} = 2\,\frac{v_1 - v_2}{c}\,\frac{\int \mu f(\vec{x},\vec{p})\,\frac{d^3x}{2}\,d^3p}{\int f(\vec{x},\vec{p})\,d^3x\,d^3p} = 2\,\frac{v_1 - v_2}{c}\,\frac{\int_0^1 \mu f \sigma v\mu\, dt\, 2\pi p^2 dp\, d\mu}{2\int_0^1 f \sigma v\mu\, dt\, 2\pi p^2 dp\, d\mu}$$

$$= \frac{2(v_1 - v_2)}{c}\,\frac{\frac{\mu^3}{3}\Big|_0^1}{\mu^2\Big|_0^1} = \frac{2}{3}\,\frac{v_1 - v_2}{c}$$

$\Rightarrow$ the relative energy gain averaged over all particles is then (for a full cycle)

$$\mathcal{E} \equiv \frac{\delta E}{E} = \frac{\delta p}{p} = \frac{4}{3}\,\frac{v_1 - v_2}{c}$$

- the escape probability is the ratio of particle flux carried by downstream flow at v_2 over the particle flux that crosses the shock front at speed c (assuming again relativistic particles)

$$P_{esc} = \frac{f_2\, 4\pi p^2 dp\, \sigma v_2\, dt}{\int_0^1 f_0\, 2\pi p^2 dp\, \sigma c\mu\, dt\, d\mu} = \frac{4v_2}{c}\,\frac{f_2}{2 f_0 \frac{\mu^2}{2}\Big|_0^1} \simeq \frac{4v_2}{c}$$

assuming that the cosmic ray distribution function is rapidly homogenised behind the shock, i.e. $f_2 = f_0$

- after n cycles, the particle has the energy

$$E = E_0 \left(1+\varepsilon \right)^n \iff n = \frac{\log \frac{E}{E_0}}{\log(1+\varepsilon)}$$

at each acceleration cycle, there is the escape probability P_e; after n cycles, the particle has the probability $(1-P_e)^n$ still to participate in the process; hence, the number of particles with energy larger than E is given by

$$N(>E) \propto \sum_{m=n}^{\infty} (1-P_e)^m = (1-P_e)^n \sum_{m=0}^{\infty} (1-P_e)^m \underset{\substack{\uparrow \\ \text{geometrical} \\ \text{series}}}{=} (1-P_e)^n \frac{1}{1-(1-P_e)} = \frac{(1-P_e)^n}{P_e}$$

using $a^{\ln b} = \left(e^{\ln a}\right)^{\ln b} = b^{\ln a}$, we can rewrite this $\left[a \equiv 1-P_e ;\ b \equiv \frac{E}{E_0} \right]$

$$N(>E) \propto \frac{1}{P_e} \left(\frac{E}{E_0}\right)^{-\tilde{\alpha}}, \quad \tilde{\alpha} = \frac{\log \frac{1}{1-P_e}}{\log(1+\varepsilon)}$$

- using (3.163) and (3.164) and Taylor expanding $\log(1+x) = x - \frac{x^2}{2} + \mathcal{O}(x^3)$, we get $\left[P_{esc} \simeq \frac{4V_2}{c} \right]$

$$\tilde{\alpha} = \frac{-\log\left(1-\frac{4V_2}{c}\right)}{\log\left(1+\frac{4}{3}\frac{V_1-V_2}{c}\right)} \approx \frac{4\frac{V_2}{c}}{\frac{4}{3}\frac{V_1-V_2}{c}} = \frac{3V_2}{V_1-V_2}$$

$$= \frac{3}{\frac{V_1}{V_2}-1} = \frac{3}{r-1}$$

where we introduced the density compression factor $r = \frac{\rho_2}{\rho_1} = \frac{V_1}{V_2}$

- the cumulative particle spectrum due to diffusive shock acceleration is

$$N(>E) = \int N(E)\,dE \propto \frac{c}{4V_2}\left(\frac{E}{E_0}\right)^{-\frac{3}{r-1}}$$

- the differential spectrum $N(E)$ has a spectral index α that connects to $\tilde{\alpha}$ via

$$-\tilde{\alpha} = 1-\alpha = -\frac{3}{r-1}$$

$$\alpha = 1 + \frac{3}{r-1} = \frac{r+2}{r-1} = 2 \quad \text{for a strong shock with } r=4$$