①

Pressure

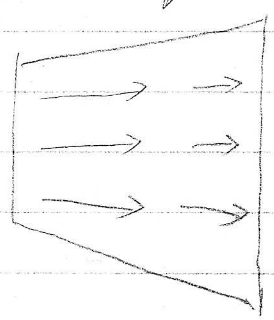
$$\rho \frac{Dv}{Dt} = \rho\left(\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v}\right) = -\nabla P + \vec{F} + \rho\nu\nabla^2\vec{v}$$

Navier-Stokes equation

dissipation

Convective acceleration

external force.



timescales: L/v for convection

L^2/ν for viscous dissipation.

$$Re = \frac{\tau_{diss}}{\tau_{adv}} = \frac{LU}{\nu}.$$

Reynolds #:

$$R_e = \frac{VL}{\nu}.$$

Incompressible turbulence:

$$\nabla \cdot u = 0, \quad u = \int v(\vec{k}, \omega) e^{ikx - \omega t} d^3k\, d\omega$$

$k \cdot v = 0 \Rightarrow$ No longitudinal disturbances (sound waves)

"Eddies"

$$R_e = \tau_{diss} / \tau_{dyn} \qquad \nu \equiv \text{viscosity}$$

$$= \frac{L^2}{\nu} \Big/ \frac{L}{U}$$

$$= \frac{UL}{\nu}$$

If $R_e \gg 1$, Dynamical growth is too fast to be stabilized by viscous dissipation

$\Rightarrow$ turbulence developes!!

③.

Injection scale L:

the size of unstable region

In steady state:

the enerny can neither accumulate on the injection scale nor dissipate visconsly

$\Rightarrow$ the only other channel for the enery transfer is through nonlinear interactions

Homework: 1(a) derive Kolmogorov Spectrum from Steady state cascade

④

Ohm's Law in rest frame

$$j' = \sigma E'$$

in Laboratory frame

$$E = \frac{j_e}{\sigma} - \frac{\vec{U}}{c} \times \vec{B},$$ together with Ampere's Law

$$\nabla \times \vec{B} = \frac{4\pi}{c} \vec{j}$$

substituted into Faraday's Law of induction

$$\frac{\partial B}{\partial t} + \nabla \times (\vec{B} \times \vec{u}) = \eta \nabla^2 B$$

where electric diffusivity $\eta = \frac{c^2}{4\pi\sigma}$.

magnetic Reynolds number

$$R_M = \frac{LU}{\eta} >> 1, \quad \eta \nabla^2 B$$

$\eta \nabla^2 B$ — magnetic diffusion is negligible

$\boxed{\text{field freezing!}}$

⑤

Anisotropy & Goldreich - Sridhar (95) Theory

3 wave interactions:

$$\vec{k}_1 + \vec{k}_2 = \vec{k}_3 \text{ ①} \quad \Rightarrow k_{\parallel,1} \text{ or } k_{\parallel,2} = 0$$

$$\omega_1 + \omega_2 = \omega_3 \text{ ②}$$

$\omega = |k_\parallel| V_A$ Cascade $\perp B_0$

In real turbulence, eq ② only needs to be satisfied within an uncertainty of $\delta\omega = 1/t_{cas}$, $\Rightarrow$ not strictly $\perp B_0$

GS95 theory:

mixing hydro motions couple to wave-like motions $\parallel B$

giving critical balance

$$\omega \sim \frac{1}{\tau_{ed}}$$

Homework 1b. Derive from the critical balance condition the scale-dependent anisotropy $k_\parallel \propto k_\perp^{2/3}$

3D energy spectrum of incompressible MHD turbulence $P(k_\perp, k_\parallel) \propto k_\perp^{-10/3} \exp\left(-\frac{k_\parallel L^{1/3}}{k_\perp^{2/3}}\right)$

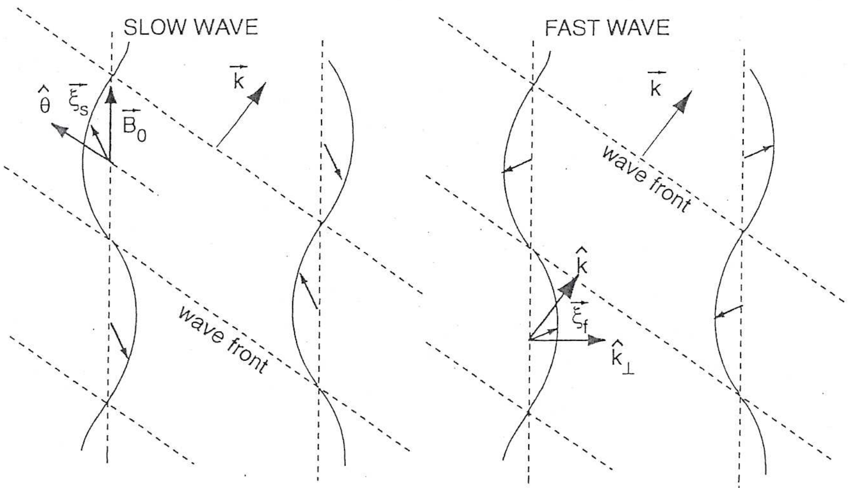


Fig. 8. Waves in real space. We show the directions of displacement vectors for a slow wave (*left*) and a fast wave (*right*). Note that $\hat{\xi}_s$ lies between $\hat{\theta}$ and $\hat{\mathrm{B}}_0$ ($= \hat{\mathrm{k}}_{\|}$) and $\hat{\xi}_f$ between $\hat{\mathrm{k}}$ and $\hat{\mathrm{k}}_{\perp}$. Again, $\hat{\theta}$ is perpendicular to $\hat{\mathrm{k}}$ and parallel to the wave front. Note also that, for the fast wave, for example, density (inferred by the directions of the displacement vectors) becomes higher where field lines are closer, resulting in a strong restoring force, which is why fast waves are faster than slow waves.

Table 1. Notations for compressible turbulence

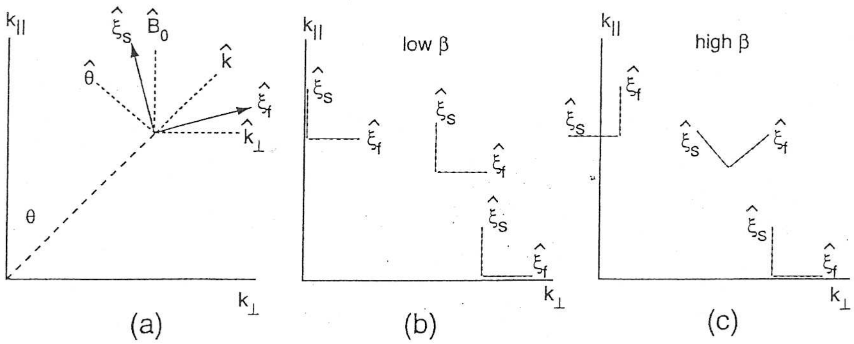


Fig. 7. (a) Directions of fast and slow basis vectors. $\hat{\xi}_f$ and $\hat{\xi}_s$ represent the directions of displacement of fast and slow modes, respectively. In the fast basis ($\hat{\xi}_f$) is always between $\hat{\mathrm{k}}$ and $\hat{\mathrm{k}}_{\perp}$. In the slow basis ($\hat{\xi}_s$) lies between $\hat{\theta}$ and $\hat{\mathrm{B}}_0$. Here, $\hat{\theta}$ is perpendicular to $\hat{\mathrm{k}}$ and parallel to the wave front. All vectors lie in the same plane formed by B_0 and k. On the other hand, the displacement vector for Alfvén waves (not shown) is perpendicular to the plane. (b) Directions of basis vectors for a very small β drawn in the same plane as in (a). The fast bases are almost parallel to $\hat{\mathrm{k}}_{\perp}$. (c) Directions of basis vectors for a very high β. The fast basis vectors are almost parallel to k. The

⑥

Compressible modes.

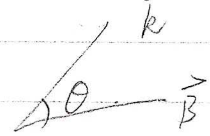
Fig. 8. Fig 7.

$$\beta = \frac{P_{gas}}{P_{mag}} = \frac{2 c_s^2}{V_A^2},$$

$\beta \gg 1$, $M_s < 1$, Alfven, slow, fast modes.

$\omega = k_{\parallel} V_A$, follow GS95 $\omega = k c_s$ decouple

$\beta \gg 1$, $M_s > 1$, not clear.



$\beta < 1$ (diffuse ISM)

slow modes, $\omega = k c_s \cos\theta$ modulated by Alfven, passive $\Rightarrow$

follow the scaling of Alfven $\begin{cases} M_A > 1 & \text{winding dynamo} \\ M_A < 1 & \end{cases}$

$$\begin{bmatrix} M_{ij}(\vec{k}) \\ K_{ij}(\vec{k}) \end{bmatrix} = \frac{\beta^2}{16} \sin^2(2\theta) \frac{k_i k_j}{k_\perp^2} k_\perp^{-10/3} \exp\left(-\frac{L^{\frac{1}{3}} k_{\parallel}}{k_\perp^{2/3}}\right) \begin{bmatrix} \cos^2\theta \\ 1 \end{bmatrix}.$$

(7)

fast modes decoupled as $\omega = kV_A$ doesn't depend on B field shearing.

3 wave resonance condition.

$$\left.\begin{array}{l}\omega_1 + \omega_2 = \omega_3 \\ \vec{k}_1 + \vec{k}_2 = \vec{k}_3\end{array}\right\} \quad \text{combined with } \omega = kV_A.$$

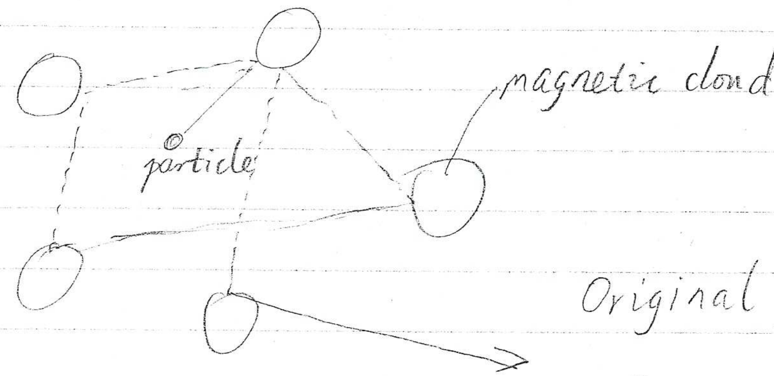
$\Rightarrow$ cascade is radial and so an isotropic energy distribution.

$$\beta < 1 \qquad \begin{bmatrix} M_{ij}(\vec{k}) \\ K_{ij}(\vec{k}) \end{bmatrix} = \frac{L^{-1/2}}{8\pi} \frac{k_i k_j}{k_\perp^2} k^{-7/2} \begin{bmatrix} \cos^2\theta \\ 1 \end{bmatrix}.$$

$$\beta > 1 \qquad = \frac{L^{-1/2}}{4\pi} \sin^2\theta \frac{k_i k_j}{k_\perp^2} k^{-7/2} \begin{bmatrix} \cos^2\theta/\beta \\ 1 \end{bmatrix}$$

(8)

page ①



Original picture of Feri acceleration

$D_{\mu\mu}$, D_{pp}, $D_{\mu p}$, $D_{\mu\mu} = Re\int_0^\infty dt \langle \dot{\mu}(t)\, \dot{\mu}^*(0) \rangle$

$$\dot{\mu} = \frac{i\varepsilon\Omega\sqrt{1-\mu^2}}{\sqrt{2}B_0}\left\{ e^{i\psi}\delta B_R[\vec{x},t] - e^{-i\psi}\delta B_L[\vec{x},t] \right\} \quad \psi = \mathrm{arccot}(k_x/k_y)$$

$$\varepsilon = \frac{|q|}{q}$$

$$\langle \delta B_i(\vec{x},t)\, \delta B_j^*(x,t) \rangle = \int d^3k\, d^3k' \langle \delta B(\vec{k},t)\delta B^*(\vec{k}',0) \rangle$$

$$= P_{ij}(\vec{k},t)\,\delta(\vec{k}-\vec{k}')$$

unperturbed orbit:

$$\otimes \quad \exp\left[i\vec{k}\cdot\vec{x}(t) - i\vec{k}'\cdot\vec{x}(0) - \omega t\right]$$

$$\begin{cases} z = z_G = v\mu t \\ x_1 = x_G - \frac{v}{\Omega}\sqrt{1-\mu^2}\sin(\Phi_0 - \Omega t) \\ y = y_G - \frac{v}{\Omega}\sqrt{1-\mu^2}\cos(\Phi_0 - \Omega t) \end{cases}$$

$$\exp(i\vec{k}\cdot\vec{x}) = \sum_{n=-\infty}^{\infty} J_n(W)\exp\left[ik_\parallel v_\parallel t + in(\psi - \phi_0 + \varepsilon\Omega t)\right]$$

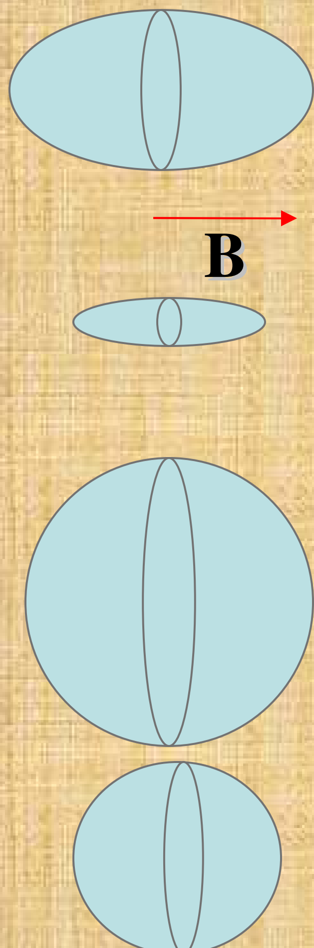
Random initial phase ϕ_0 and ψ $\qquad W = \frac{k_\perp v_\perp}{\Omega}$

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi_0\, e^{i(n-m)} = \delta_{nm}$$

Anisotropy of MHD modes

Alfven and slow modes (GS95)

fast modes



Equal velocity correlation contour (Cho & Lazarian 02)