Graduate mini couse:
“High Energy Astrophysics – Selected Topics”

Assignment #1

Due 1pm, Thu Mar 6

This assignment is due before class on Mar 5; that is, at 1:00pm **sharp**. Assignments may be handed in in person at class; may be placed at our offices 1303 (Huirong) or 1304 (Christoph). The instructors will have office hours 2-3pm on Wed.

Show your work, and good luck!

Question 1 - Generalized Force Term due to Turbulence [10 pts]

(10 pts) As we discussed in class, the appropriate form of the *relativistic Vlasov equation* reads as follows

$$\frac{\partial f}{\partial t} + v\mu \frac{\partial f}{\partial Z} - \varepsilon\Omega \frac{\partial f}{\partial \phi} + \frac{1}{p^2}\frac{\partial}{\partial x_\sigma}(p^2 g_{x_\sigma} f) = S(\boldsymbol{x}, \boldsymbol{p}, t), \tag{1}$$

where $f = f(\boldsymbol{x}, \boldsymbol{p}, t)$ represents the phase space density of plasma particles of a given sort, $\boldsymbol{p} = \gamma m \boldsymbol{v}$ is the momentum, x_σ represents the coordinate set $x_\sigma = (p, \mu, \phi, X, Y, Z)$ with spherical coordinates in momentum space and the coordinates of the guiding center for gyration $\boldsymbol{R} = (X, Y, Z)$, and Z denotes the spatial variable along the locally uniform magnetic field $\boldsymbol{B}_0 = B_0\, \boldsymbol{e}_z$ implying $\mu = p_\parallel / p$. The generalized force term g_σ includes the effects of the randomly fluctuating electromagnetic fields and S denotes the sources and sinks of particles.

Compute the generalized force terms g_p and g_μ solely as functions of our new coordinate system x_σ by introducing the left-handed and right-handed polarized field components,

$$\delta B_{\mathrm{L,R}} \equiv \frac{1}{\sqrt{2}}(\delta B_x \pm i\, \delta B_y), \quad \delta B_\parallel \equiv \delta B_z,$$

$$\delta E_{\mathrm{L,R}} \equiv \frac{1}{\sqrt{2}}(\delta E_x \pm i\, \delta E_y), \quad \delta E_\parallel \equiv \delta E_z.$$

Question 2 - Quasilinear Theory of Cosmic Rays [20 pts]

2 (a)

(15 pts) In class, we have shown that under certain conditions the relativistic Vlasov equation corresponds to the Fokker-Planck equation that reads for the gyrotropic phase space density $f(Z, p, \mu, t)$ as follows:

$$\frac{\partial f}{\partial t} + v\mu\frac{\partial f}{\partial Z} - S_0(p,\mu,t) = \frac{\partial}{\partial \mu}\left[D_{\mu\mu}\frac{\partial f}{\partial \mu} + D_{\mu p}\frac{\partial f}{\partial p}\right] + \frac{1}{p^2}\frac{\partial}{\partial p}p^2\left[D_{\mu p}\frac{\partial f}{\partial \mu} + D_{pp}\frac{\partial f}{\partial p}\right], \tag{2}$$

where we only consider isotropic source terms $S_0(p, \mu, t)$ and the Fokker-Planck coefficients $D_{\mu\mu}, D_{\mu p}, D_{pp}$ have been defined in class. Your task is to derive the diffusion convection equation for cosmic ray transport,

$$\frac{\partial F}{\partial t} - S_0(Z,p,t) \quad = \quad \frac{\partial}{\partial Z}\left[\kappa(Z,p,t)\frac{\partial F}{\partial Z}\right] + \frac{1}{p^2}\frac{\partial}{\partial p}\left[p^2\Gamma(Z,p,t)\frac{\partial F}{\partial p}\right] + \frac{v}{4}\frac{\partial A_1}{\partial Z}\frac{\partial F}{\partial p} - \frac{1}{4p^2}\frac{\partial(p^2 v A_1)}{\partial p}\frac{\partial F}{\partial Z}, \tag{3}$$

where the spatial diffusion coefficient κ, the momentum diffusion coefficient Γ, and A_1 are determined by pitch angle averages of three Fokker-Planck coefficients

$$\begin{aligned} \kappa &= \frac{v^2}{8}\int_{-1}^{1} \mathrm{d}\mu \, \frac{(1-\mu^2)^2}{D_{\mu\mu}(\mu)}, \\ \Gamma &= \frac{1}{2}\int_{-1}^{1} \mathrm{d}\mu \left[D_{pp}(\mu) - \frac{D_{\mu p}^2(\mu)}{D_{\mu\mu}(\mu)} \right], \\ A_1 &= \int_{-1}^{1} \mathrm{d}\mu \, (1-\mu^2)\, \frac{D_{\mu p}(\mu)}{D_{\mu\mu}(\mu)}. \end{aligned} \tag{4}$$

We derived this diffusion-convection equation in the frame that is comoving with the plasma. By applying a Galilean transformation to Eqn. (3), show that it reduces to the well-known form in the laboratory system:

$$\frac{\partial F}{\partial t} - S_0(Z,p^*,t) = \frac{\partial}{\partial Z}\left[\kappa(Z,p^*,t)\frac{\partial F}{\partial Z}\right] - V\frac{\partial F}{\partial Z} + \frac{1}{p^{*2}}\frac{\partial}{\partial p^{*2}}\left[p^{*2}\Gamma(Z,p^*,t)\frac{\partial F}{\partial p^*}\right] + \frac{p^*}{3}\frac{\partial V}{\partial Z}\frac{\partial F}{\partial p^*}, \tag{5}$$

where we introduced the plasma bulk speed U, and the cosmic ray bulk speed V,

$$V = v^* + \frac{1}{4p^{*2}}\frac{\partial(p^{*2}v^*A_1)}{\partial p^*}, \quad A_1 = \int_{-1}^{1} \mathrm{d}\mu\,(1-\mu^2)\,\frac{D_{\mu p}(\mu)}{D_{\mu\mu}(\mu)}. \tag{6}$$

Here, we use a mixed coordinate system in which the space coordinates ($\boldsymbol{X}$) are measured in the lab system and the momentum coordinates (p^*, μ^*) in the rest frame of the streaming plasma.

Procedure: Split the total density f into an isotropic part F and an anisotropic part δf,

$$f(Z,p,\mu,t) = F(z,p,t) + \delta f(Z,p,\mu,t) \quad \text{where} \quad F(z,p,t) \equiv \frac{1}{2}\int_{-1}^{1}\mathrm{d}\mu\, f(Z,p,\mu,t), \tag{7}$$

and use similar ideas as in class to combine the pitch-angle averaged equation with the original one. Use then the diffusive approximation which applies if the isotropic particle density is slowly evolving, i.e.

$$\frac{\partial F}{\partial t} = \mathcal{O}\left(\frac{F}{T}\right), \quad \frac{\partial F}{\partial z} = \mathcal{O}\left(\frac{F}{L}\right)$$

with typical length scales $L \gg \lambda$ and time scales $T \gg \tau$ much larger than the mean free path $\lambda = v\tau$ and the pitch angle scattering relaxation time $\tau \simeq \mathcal{O}(1/D_{\mu\mu})$. Under these conditions, the particles can reach locally near-equilibrium which results in a small anisotropy, i.e. $\delta f \ll F$. If you regard δf of order τ and F of order 1, and recall that $D_{\mu p}$ and D_{pp} are of order ε/τ and ε^2/τ, respectively, where $\varepsilon = v_{\mathrm{ph}}/v = v_A/v \ll 1$, smaller than $D_{\mu\mu} = \mathcal{O}(1/\tau)$, you may characterize the differential operators by different timescales. To lowest order, the approximate equation should look as follows:

$$\frac{\partial}{\partial\mu}\left[D_{\mu\mu}\frac{\partial\delta f}{\partial\mu} + D_{\mu p}\frac{\partial F}{\partial p}\right] \simeq v\mu\frac{\partial F}{\partial Z}. \tag{8}$$

Solve this equation for δf, determine the integration constants appropriately, and insert the solution into the pitch-angle averaged equation enables you to derive Eqns. (3) and (4). To perform the Galilean transformation, use a mixed coordinate system in which the space coordinates ($\boldsymbol{X}'$) are measured in the lab system and the momentum coordinates (p^*, μ^*) in the rest frame of the streaming plasma. Justify and use the following system of equations for your transformation:

$$t = t', \quad X = X', \quad Y = Y', \quad Z = Z' + Ut' \tag{9}$$

$$p^* = p^*\left(1 - \frac{t'}{3}\frac{\partial U}{\partial Z'}\right), \quad \mu^* = \mu. \tag{10}$$

Finally, in your derivation, you might want to use the general scaling of the rate of adiabatic deceleration,

$$A_1(Z,p) = A_1^0(Z)\frac{p}{v}. \tag{11}$$

2 (b)

(5 pts) Explain in your own words the relevant steps and approximations that were necessary in deriving the diffusion convection equation for cosmic ray transport (5) from the Vlasov equation (1).