Graduate mini couse: "High Energy Astrophysics – Selected Topics"

Assignment #2

Due 1pm, Thu Mar 13

This assignment is due before class on Mar 13; that is, at 1:00pm **sharp**. Assignments may be handed in in person at class; may be placed at our offices 1303 (Huirong) or 1304 (Christoph). The instructors will have office hours 2-3pm on Wed.

Show your work, and good luck!

Question 1 - scaling relation of turbulence [10 pts]

1 (a)

(3 pts) Derive Kolmogorov spectrum $E_k \propto k^{-5/3}$ for hydrodynamic turbulence for the steady state cascade.

1 (b)

(7 pts) In class, we have shown that strong MHD turbulence satisfy the critical balance condition $\omega = 1/\tau_{edd}$ in Goldreich-Sridhar theory. Derive from that the anisotropy $k_\parallel \sim k_\perp^{2/3} L^{-1/3}$ for the turbulence.

Question 2 - Fokker Planck coefficients for Cosmic Ray transport [20 pts]

2 (a)

(8 pts) From

$$\begin{aligned}
\bar{g}_\mu &= \frac{\epsilon\Omega\sqrt{1-\mu^2}}{B_0^2} \sum_{n=-\infty}^{n=\infty} \int_{-\infty}^{\infty} d^3k \exp[in(\psi-\phi_0) + i(k_\parallel v_\parallel + n\epsilon\Omega)t] \\
& \frac{i}{\sqrt{2}} \left[B_R(\mathbf{k},t) e^{i\psi} J_{n+1}(W) - B_L(\mathbf{k},t) e^{i\psi} J_{n-1}(W) \right], \qquad (1)
\end{aligned}$$

where $W = \frac{k_\perp v_\perp}{\Omega}$, derive the general expressions for Fokker-Planck coeffcients:

$$\begin{aligned}
\begin{pmatrix} D_{\mu\mu} \\ D_{pp} \end{pmatrix} &= \frac{\pi\Omega^2(1-\mu^2)}{2} \int_{k_{min}}^{k_{max}} d^3k \delta(k_\parallel v_\parallel - \omega \pm \epsilon\Omega) \begin{pmatrix} 1 \\ m^2 v_A^2 \end{pmatrix} \\
&\times \left\{ [J_2^2(W) + J_0^2(W)] \begin{bmatrix} M_{RR}(\mathbf{k}) + M_{LL}(\mathbf{k}) \\ K_{RR}(\mathbf{k}) + K_{LL}(\mathbf{k}) \end{bmatrix} \right. \\
& \left. -2J_2(W)J_0(W) \left[e^{i2\psi} \begin{pmatrix} M_{RL}(\mathbf{k}) \\ K_{RL}(\mathbf{k}) \end{pmatrix} + e^{-i2\psi} \begin{pmatrix} M_{LR}(\mathbf{k}) \\ K_{LR}(\mathbf{k}) \end{pmatrix} \right] \right\}, \qquad (2)
\end{aligned}$$

where $M_{ij}(\mathbf{k}) = \frac{<B_i(\mathbf{k})B_j^*(\mathbf{k})>}{B_0^2}$, $K_{ij}(\mathbf{k}) = \frac{<v_i(\mathbf{k})v_j^*(\mathbf{k})>}{v_A^2}$.

2 (b)

(12 pts) Derive the Fokker-Planck coefficients for scattering of cosmic rays by the Alfvénic turbulence

$$\begin{pmatrix} D_{\mu\mu} \\ D_{pp} \end{pmatrix} = \begin{pmatrix} 1 \\ m^2 v_A^2 \end{pmatrix} \frac{v^{2.5}\mu^{5.5}}{\Omega^{1.5} L^{2.5} \sqrt{1-\mu^2}} \Gamma[6.5, k_{max}^{-2/3} |k_{\parallel,res}| L^{1/3}].$$

Hint: As shown in class, the correlation tensor for the Alfvénic turbulence is

$$S_{ij} = \frac{B_0^2}{6\pi L^{1/3}} \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right) k_\perp^{-10/3} \exp\left(-\frac{|k_\parallel L^{1/3}|}{k_\perp^{2/3}} \right).$$