A pedagogical introduction to cosmic rays, magnetic fields and galaxy formation

Part 1: Magnetic fields

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Outline of the topics of the four lectures:

- **Magnetic fields**
  - * Properties and observables of astrophysical magnetic fields
  - * Generation and evolution of magnetic fields
- **Cosmic ray acceleration and observables**
  - * Properties of Galactic cosmic rays
  - * Cosmic ray acceleration by shocks and turbulence
- **Cosmic ray transport and non-thermal emission**
  - * Cosmic ray transport and particle-wave interactions
  - * Non-thermal emission processes from radio to gamma rays
- **The physics of galaxy formation**
  - * Puzzles in galaxy formation
  - * Feedback by stars and active galactic nuclei

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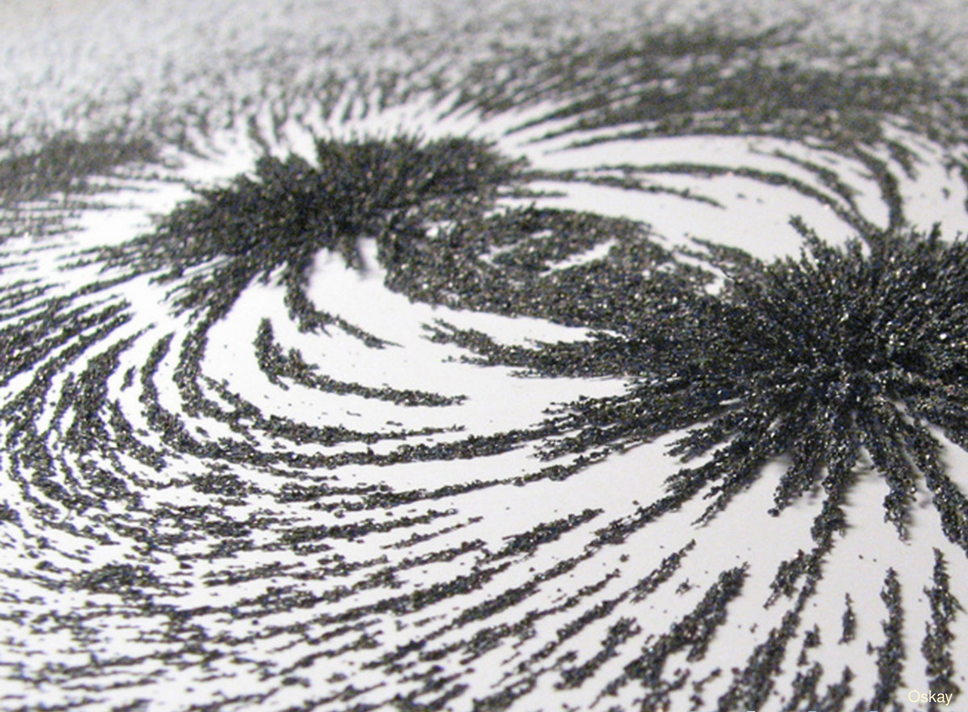
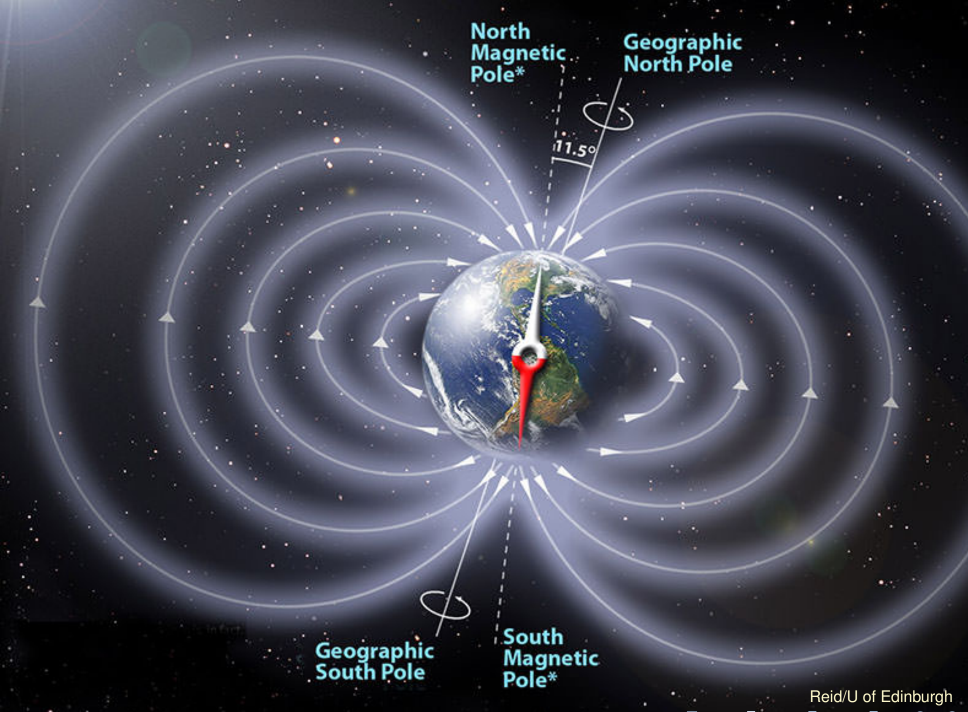
- magnetic fields are essential for accelerating cosmic rays (CRs): diffusive shock acceleration (1^{st} order Fermi), turbulent magneto-hydrodynamic (MHD) interactions with CRs (2^{nd} order Fermi)
- magnetic fields trace violent high-energy astrophysical processes by illuminating distant CR electron populations through synchrotron emission: structure formation shocks, supernovae, . . .

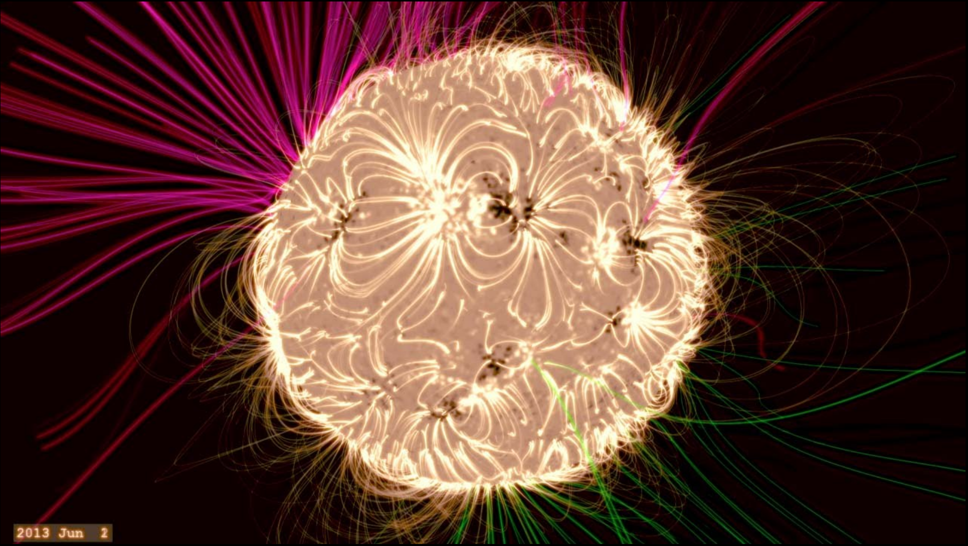


Oskay



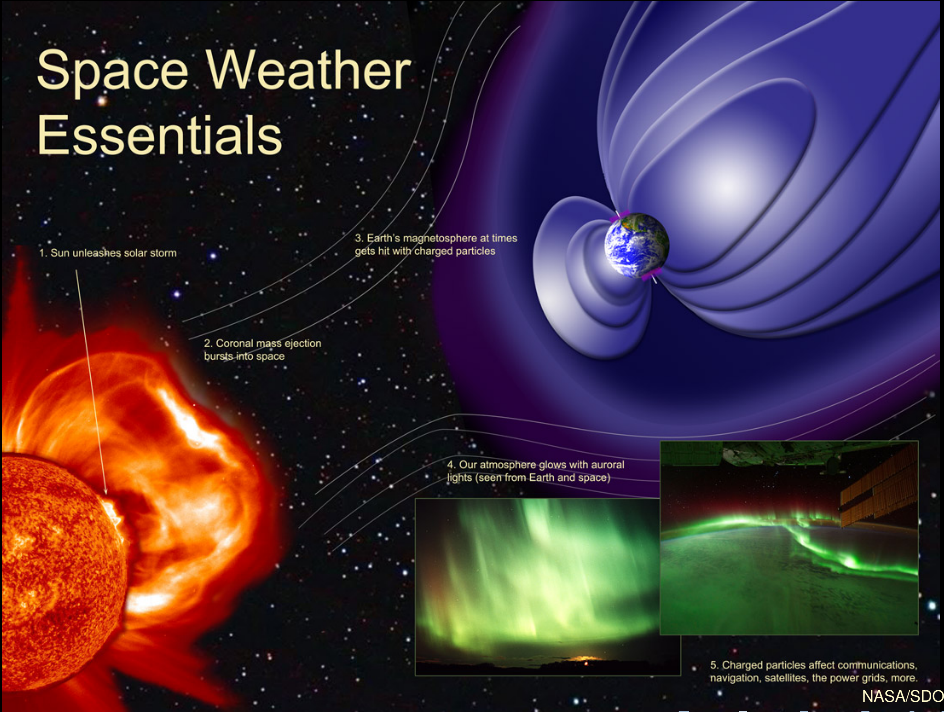
Reid/U of Edinburgh

Model of the magnetic field of the Sun

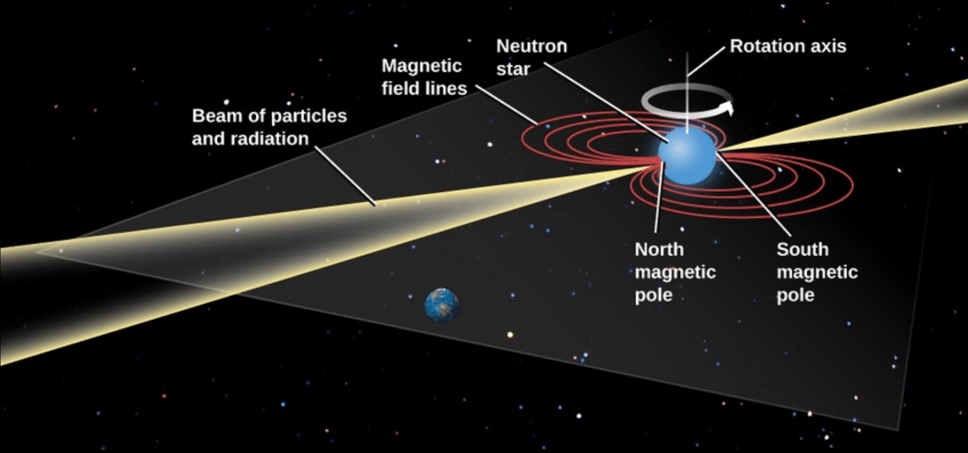


NASA/Duberstein

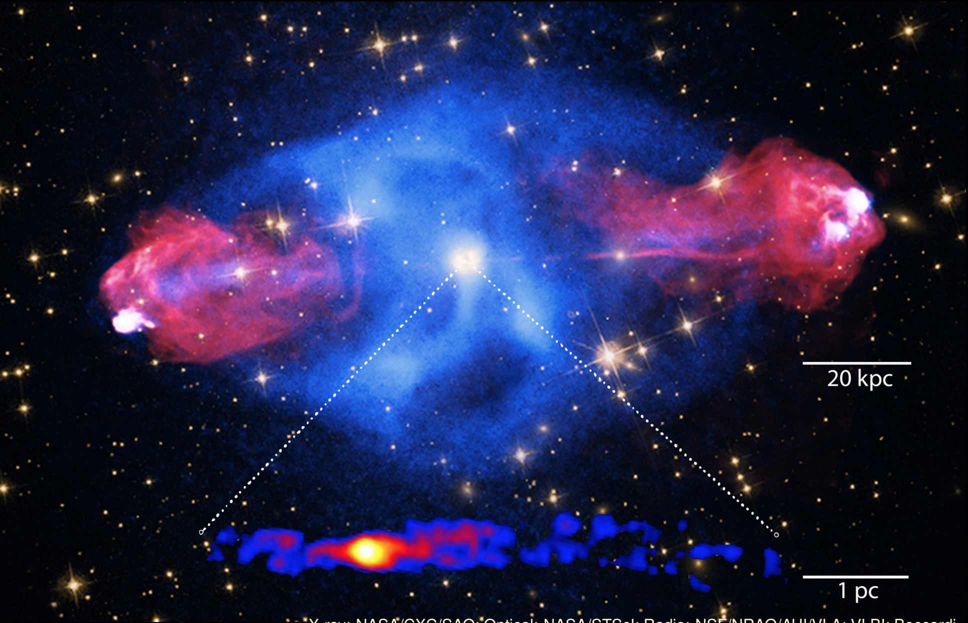
Space Weather Essentials



Model of a pulsar or magnetar

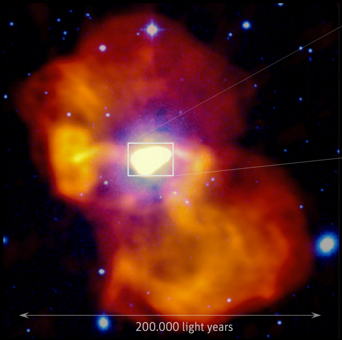


Synchrotron emission of the Cygnus A jet (radio)

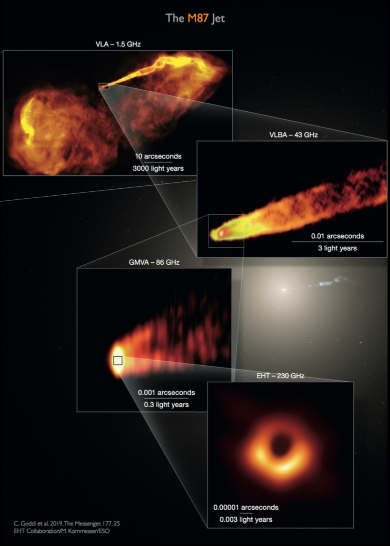


X-ray: NASA/CXC/SAO; Optical: NASA/STScI; Radio: NSF/NRAO/AUI/VLA; VLBI: Boccardi

The black hole in the center of the Virgo galaxy cluster

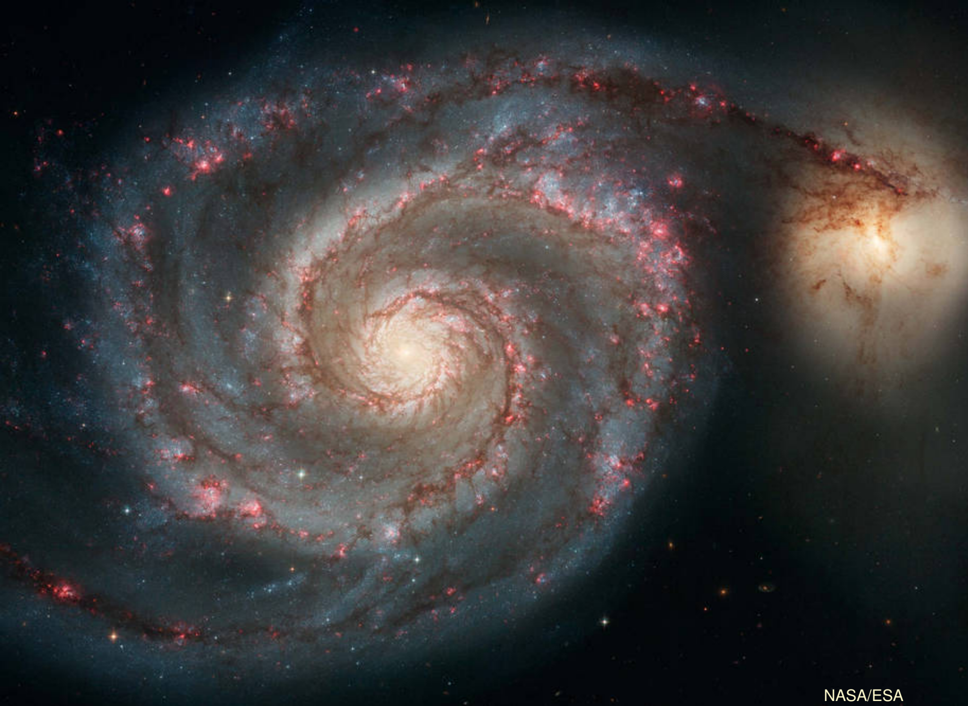


LOFAR/de Gasperin



Messier 87: the black hole in polarized light

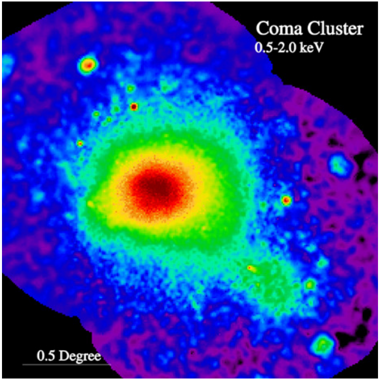
Event Horizon Collaboration


NASA/ESA


NASA/ESA/SOFIA

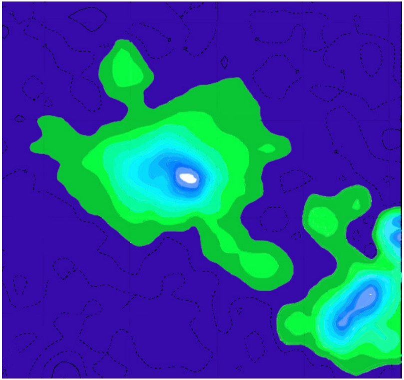
Magnetic field on the largest scales

Giant radio halo in the Coma galaxy cluster



thermal X-ray emission

Snowden/MPE/ROSAT

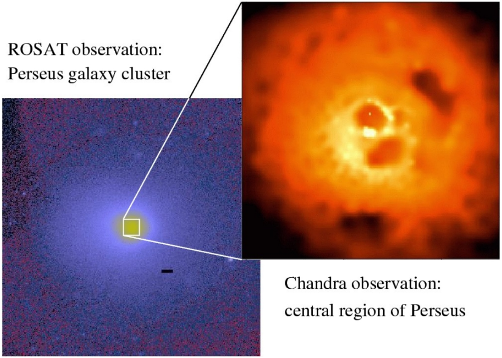


radio synchrotron emission

Deiss/Effelsberg

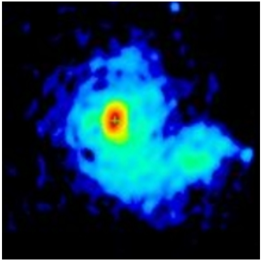
Magnetic field on the largest scales

Radio mini halo in the Perseus galaxy cluster



thermal X-ray emission

ROSAT; NASA/IoA/Fabian



radio synchrotron emission

Pedlar/VLA

Diffuse radio phenomena in galaxy clusters

- radio halos:
  centrally located, more regular morphology, unpolarized
  - giant radio halos: occur in merging clusters, $>$ 1 Mpc-sized, morphology similar to X-rays
  - radio mini halos: occur in cool core clusters, few times 100 kpc in size, emission extends over cool core

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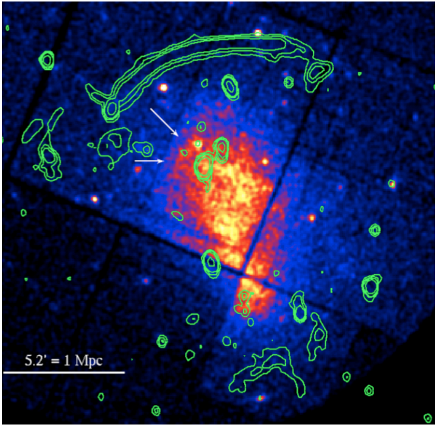
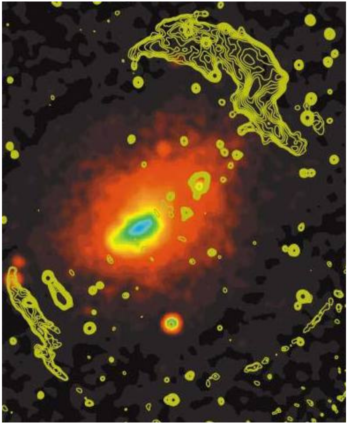
Radio shock: double relic sources



CIZA J2242.8+5301 (“sausage relic”)

X-ray: XMM; radio: WSRT/Ogrean



Abell 3667

radio: Johnston-Hollitt; X-ray: ROSAT/PSPC

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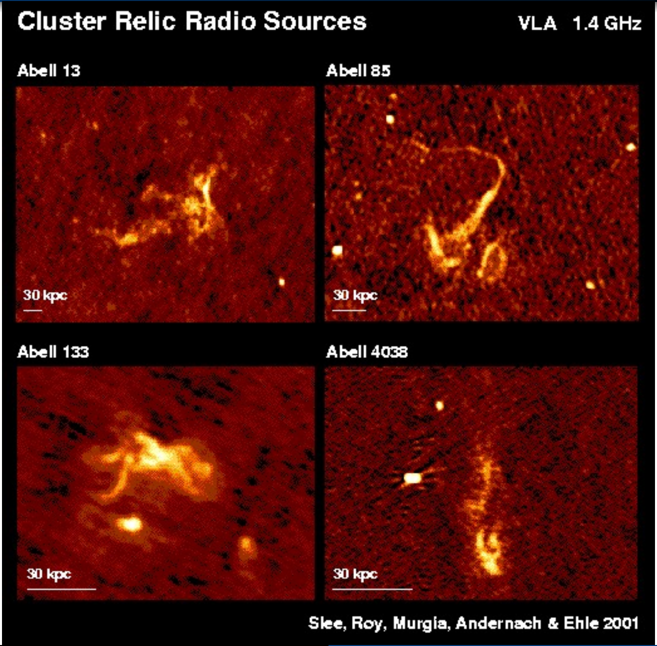
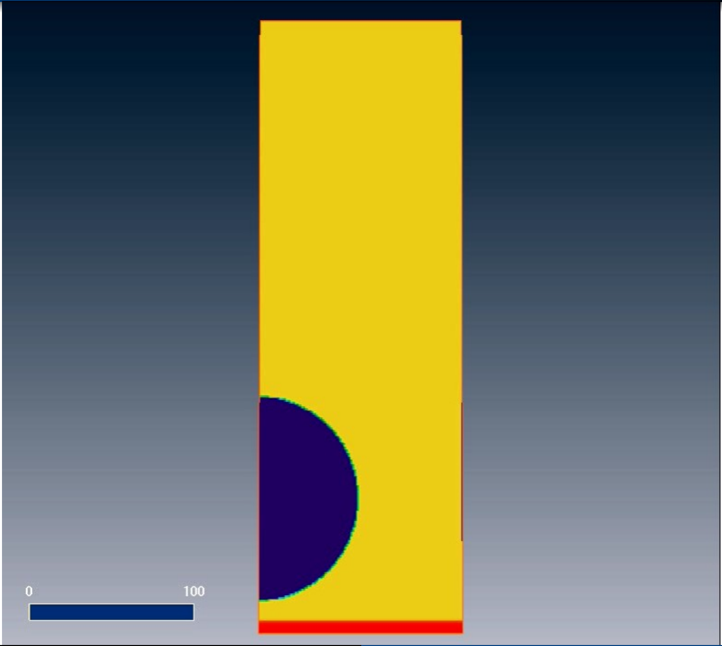
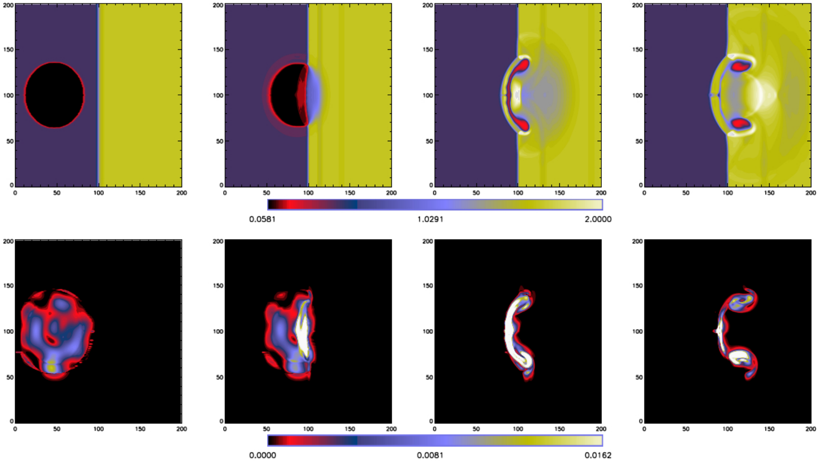
Radio phoenix



Shock overruns an aged radio bubble (Pfrommer & Jones 2011)



Bubble transformation to vortex ring



Enßlin & Brüggen (2002): gas density *(top)* and magnetic energy density *(bottom)*

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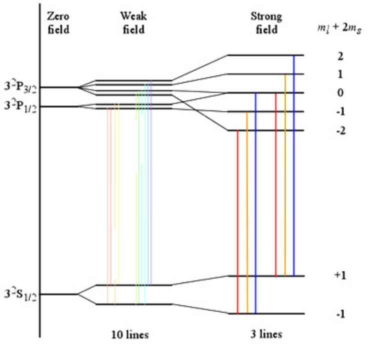
- radio relics:
  irregular morphology, polarized
  - radio shock: at cluster periphery ($<$ Mpc), in some cases coincident with weak X-ray shock, polarized $\rightarrow$ *diffusive shock acceleration (1st order Fermi)*
  - radio relic bubble: aged radio cocoon, steep spectrum
  - radio phoenix: shock-revived bubble that has already faded out of the radio window $\rightarrow$ *adiabatic compression*

Zeeman effect:

- The Zeeman effect describes the splitting of an atomic level and hence the associated spectral line into several components in the presence of a static magnetic field.
- The amount of splitting depends on the strength of the magnetic field. The splitting is associated with the quantum number of the orbital angular momentum.
- Detection requires high spectral resolution and sources of high densities (stars, cores of molecular clouds, . . .).

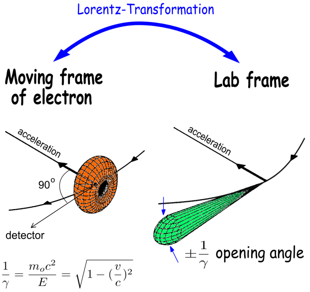


UCL/Jones

Observing magnetic fields in astrophysics – 2

Synchrotron emission:

- Charged particles emit electromagnetic radiation when accelerated, e.g. due to the Lorentz force of a magnetic field.
- This emission is axisymmetric with respect to the acceleration direction in the *particle's rest frame*.
- If the particles move relativistically, then the emission in the *lab frame* is beamed into a forward cone of an opening angle $\theta \sim \gamma^{-1}$ (where γ is the Lorentz factor).



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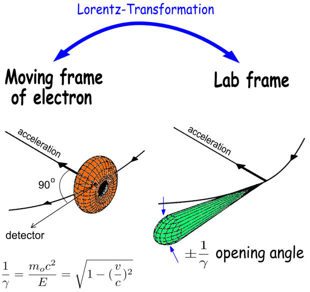
- If the particles move relativistically, then the emission in the *lab frame* is beamed into a forward cone of an opening angle $\theta \sim \gamma^{-1}$ (where γ is the Lorentz factor).
- Because the emission (= transverse electromagnetic wave) propagates in a narrow cone, it is linearly polarized.
- The typical synchrotron frequency is

$$\nu_{\mathrm{synch}} = \frac{3eB}{2\pi\, m_{\mathrm{e}} c}\,\gamma^2 \simeq 1\ \mathrm{GHz}\,\frac{B}{\mu\mathrm{G}}\left(\frac{\gamma}{10^4}\right)^2 .$$

- Power-law cosmic ray electron momentum distributions imply power-law (radio) synchrotron spectra.



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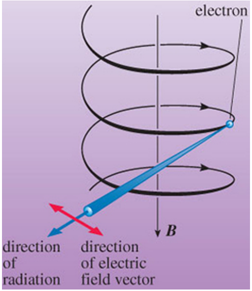
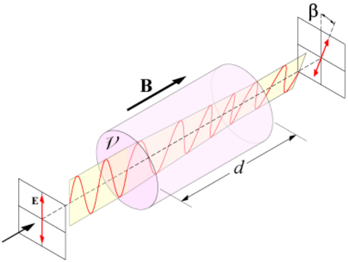


Open University

Faraday rotation:

- Faraday rotation describes rotation of a linearly polarized electro-magnetic wave in the presence of a line-of-sight magnetic field because of the birefringent property of a plasma.
- This can be seen by splitting the linearly polarized wave into right- and left-hand circularly polarized waves, which propagate at slightly different speeds.
- The observed polarization angle ϕ_{obs} is modified from its intrinsic position angle, $\phi_{\mathrm{intrinsic}}$.



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Observing magnetic fields in astrophysics – 3

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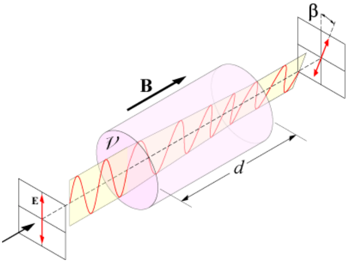
- The observed polarization angle ϕ_{obs} is modified from its intrinsic position angle, $\phi_{\text{intrinsic}}$.
- The rate of rotation scales with the wavelength squared and is given by

$$\phi_{\text{obs}}(\boldsymbol{x}_\perp) = \lambda^2 \text{RM}(\boldsymbol{x}_\perp) + \phi_{\text{intrinsic}}(\boldsymbol{x}_\perp),$$

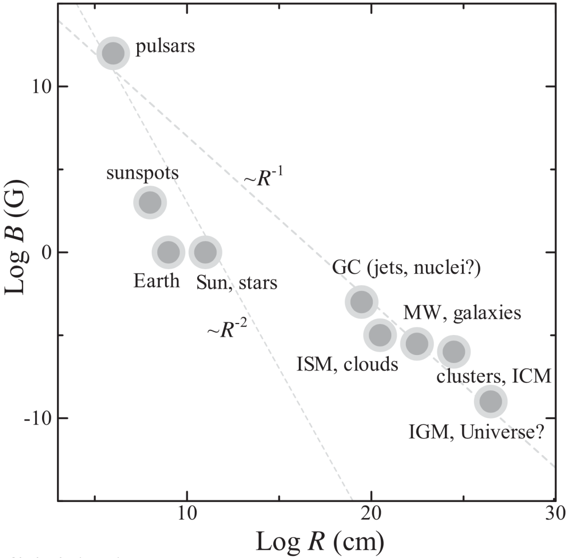
$$\text{RM}(\boldsymbol{x}_\perp) = \frac{e^3}{2\pi\, m_e^2 c^4} \int_0^d n_e(\boldsymbol{x}_\perp, l)\, \boldsymbol{B}\cdot \mathrm{d}\boldsymbol{l}$$

$$= 812\, \frac{\text{rad}}{\text{m}^2}\, \frac{B}{\mu\text{G}}\, \frac{n_e}{10^{-3}\text{cm}^{-3}}\, \frac{d}{\text{Mpc}}.$$



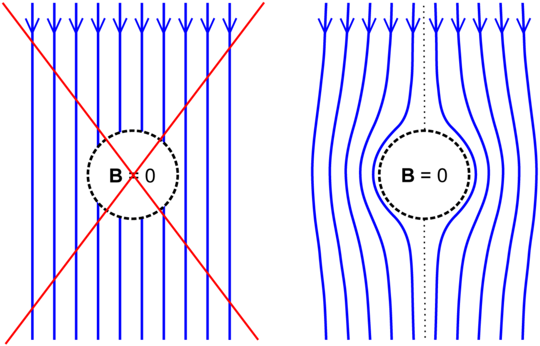
Cosmic magnetic fields

From the strongest to weakest field strengths and from compact to diffuse sources



Akahori+ (2018)

Geometrical meaning of $\nabla \cdot \boldsymbol{B} = 0$



Spruit (2013)

- There are no magnetic sources: hence, magnetic field lines “have no ends.”
- Magnetic field wraps around field-free regions and cannot end at the boundaries.
- Magnetic field forms closed loops or extends to infinity (i.e., the loop is closed at infinity).

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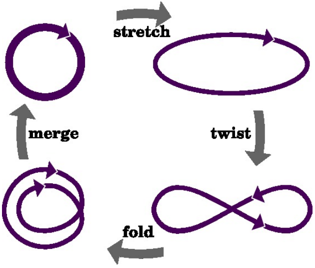

Zel'dovich+ (1983), sketch: Schober

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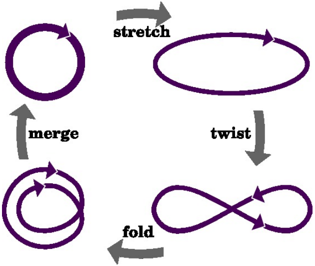
- ***Saturation.*** Field growth stops at a sizeable fraction of the turbulent energy when magnetic forces become strong enough to resist the stretching and folding motions



Zel'dovich+ (1983), sketch: Schober

The Biermann battery – 1

- Electron and proton momenta change due to the Lorentz force, the pressure and viscous forces:

$$m_\mathrm{e}\frac{\mathrm{d}\boldsymbol{v}_\mathrm{e}}{\mathrm{d}t} = -e\left(\boldsymbol{E} + \frac{\boldsymbol{v}_\mathrm{e}}{c}\times\boldsymbol{B} + \frac{1}{en_\mathrm{e}}\boldsymbol{\nabla}P_\mathrm{e}\right) - \frac{\nu_\mathrm{visc}m_\mathrm{e}}{n_\mathrm{e}}(\boldsymbol{v}_\mathrm{e} - \boldsymbol{v}_\mathrm{p}),$$

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- If $T_\mathrm{p} = T_\mathrm{e}$, we can neglect the proton equation because protons move on average slower than electrons by a factor $\sqrt{m_\mathrm{p}/m_\mathrm{e}}$.
- Viscous forces are very small on large scales: we drop the term $\propto \nu_\mathrm{visc}$.
- We assume a steady state (i.e., $\tau > \omega_\mathrm{pl}^{-1}$, where $\omega_\mathrm{pl}^2 = 4\pi n_\mathrm{e}e^2/m_\mathrm{e}$ is the plasma frequency) and solve for $\boldsymbol{E}$:

$$\boldsymbol{E} = -\frac{\boldsymbol{v}_\mathrm{e}\times\boldsymbol{B}}{c} - \frac{\boldsymbol{\nabla}P_\mathrm{e}}{en_\mathrm{e}}.$$

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$$\boldsymbol{E} = -\frac{\boldsymbol{v}_{\rm e}\times\boldsymbol{B}}{c} - \frac{\boldsymbol{\nabla} P_{\rm e}}{en_{\rm e}}.$$

- Multiplying this equation by $-c$, taking the curl of it and using Faraday's law, we obtain

$$\frac{\partial\boldsymbol{B}}{\partial t} = -c\boldsymbol{\nabla}\times\boldsymbol{E} = \boldsymbol{\nabla}\times(\boldsymbol{v}_{\rm e}\times\boldsymbol{B}) + \frac{c}{e}\boldsymbol{\nabla}\times\left(\frac{\boldsymbol{\nabla} P_{\rm e}}{n_{\rm e}}\right). \qquad (1)$$

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The Biermann battery – 2

- Using $P_\mathrm{e} = n_\mathrm{e} k_\mathrm{B} T_\mathrm{e}$ and the identities $\boldsymbol{\nabla} \times (f \boldsymbol{\nabla} g) \equiv \boldsymbol{\nabla} f \times \boldsymbol{\nabla} g$ and $\boldsymbol{\nabla} \times \boldsymbol{\nabla} f \equiv \mathbf{0}$, we can rewrite the second term of Eq. (1):

$$
\begin{aligned}
\frac{1}{k_\mathrm{B}} \boldsymbol{\nabla} \times \left(\frac{\boldsymbol{\nabla} P_\mathrm{e}}{n_\mathrm{e}} \right) &= \boldsymbol{\nabla} \times \left[\frac{1}{n_\mathrm{e}} \boldsymbol{\nabla} (n_\mathrm{e} T_\mathrm{e}) \right] = \boldsymbol{\nabla} \times (\boldsymbol{\nabla} T_\mathrm{e}) + \boldsymbol{\nabla} \times \left(\frac{T_\mathrm{e}}{n_\mathrm{e}} \boldsymbol{\nabla} n_\mathrm{e} \right) \\
&= \boldsymbol{\nabla} \left(\frac{T_\mathrm{e}}{n_\mathrm{e}} \right) \times \boldsymbol{\nabla} n_\mathrm{e} = \frac{1}{n_\mathrm{e}} \boldsymbol{\nabla} T_\mathrm{e} \times \boldsymbol{\nabla} n_\mathrm{e} - \frac{T_\mathrm{e}}{n_\mathrm{e}^2} \boldsymbol{\nabla} n_\mathrm{e} \times \boldsymbol{\nabla} n_\mathrm{e} \\
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\end{aligned}
$$

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\end{aligned}
$$

- Hence, we obtain the Biermann battery equation,

$$
\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{v}_\mathrm{e} \times \boldsymbol{B}) - \frac{c k_\mathrm{B}}{e n_\mathrm{e}} \boldsymbol{\nabla} n_\mathrm{e} \times \boldsymbol{\nabla} T_\mathrm{e}.
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The Biermann battery – 2

- Using $P_\mathrm{e} = n_\mathrm{e} k_\mathrm{B} T_\mathrm{e}$ and the identities $\boldsymbol{\nabla} \times (f \boldsymbol{\nabla} g) \equiv \boldsymbol{\nabla} f \times \boldsymbol{\nabla} g$ and $\boldsymbol{\nabla} \times \boldsymbol{\nabla} f \equiv \mathbf{0}$, we can rewrite the second term of Eq. (1):

$$\frac{1}{k_\mathrm{B}} \boldsymbol{\nabla} \times \left(\frac{\boldsymbol{\nabla} P_\mathrm{e}}{n_\mathrm{e}} \right) = \boldsymbol{\nabla} \times \left[\frac{1}{n_\mathrm{e}} \boldsymbol{\nabla} (n_\mathrm{e} T_\mathrm{e}) \right] = \boldsymbol{\nabla} \times (\boldsymbol{\nabla} T_\mathrm{e}) + \boldsymbol{\nabla} \times \left(\frac{T_\mathrm{e}}{n_\mathrm{e}} \boldsymbol{\nabla} n_\mathrm{e} \right)$$
$$= \boldsymbol{\nabla} \left(\frac{T_\mathrm{e}}{n_\mathrm{e}} \right) \times \boldsymbol{\nabla} n_\mathrm{e} = \frac{1}{n_\mathrm{e}} \boldsymbol{\nabla} T_\mathrm{e} \times \boldsymbol{\nabla} n_\mathrm{e} - \frac{T_\mathrm{e}}{n_\mathrm{e}^2} \boldsymbol{\nabla} n_\mathrm{e} \times \boldsymbol{\nabla} n_\mathrm{e}$$
$$= \frac{1}{n_\mathrm{e}} \boldsymbol{\nabla} T_\mathrm{e} \times \boldsymbol{\nabla} n_\mathrm{e}.$$

- Hence, we obtain the Biermann battery equation,

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{v}_\mathrm{e} \times \boldsymbol{B}) - \frac{c k_\mathrm{B}}{e n_\mathrm{e}} \boldsymbol{\nabla} n_\mathrm{e} \times \boldsymbol{\nabla} T_\mathrm{e}.$$

- This equation shows that if there is no magnetic field to start with (i.e., a vanishing first term on the right-hand side), then the magnetic field can be generated by a baroclinic flow with $\boldsymbol{\nabla} n_\mathrm{e} \times \boldsymbol{\nabla} T_\mathrm{e} \neq \mathbf{0}$.

AIP

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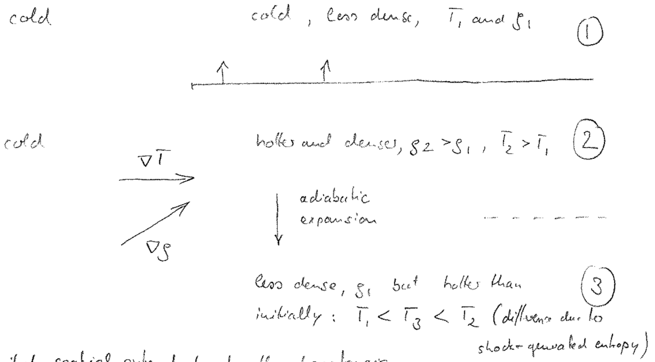
- This could be achieved in shocks of the interstellar medium, in ionization fronts, or similar astrophysical sites; in general, baroclinic flows are sourced by rotational motions at shocks of finite extent such as the chaotic collapse of a proto-galaxy.

AIP

The Biermann battery – 3

- Consider a shock of finite extent that propagates into zero-pressure medium.



⇒ shocks of limited spatial extent break the barotropic relation $P_e = P_e(n_e)$ which couples P_e to n_e

The Biermann battery – 4

- Magnetic fields generated through this process have very small field strengths: adopting a characteristic density and temperature gradient length of L of a proto-galaxy and assuming gravitational collapse on the free-fall time, $\tau \sim 1/\sqrt{G\rho}$, we obtain

$$B \sim \frac{ck_\mathrm{B} T_\mathrm{e}}{e} \frac{\tau}{L^2} \sim \frac{ck_\mathrm{B} T_\mathrm{e}}{e} \frac{1}{\sqrt{G\rho}L^2}$$

$$\sim 10^{-20}\mathrm{G} \left(\frac{T_\mathrm{e}}{10^4\,\mathrm{K}}\right) \left(\frac{n}{1\,\mathrm{cm}^{-3}}\right)^{-1/2} \left(\frac{L}{\mathrm{kpc}}\right)^{-2}.$$

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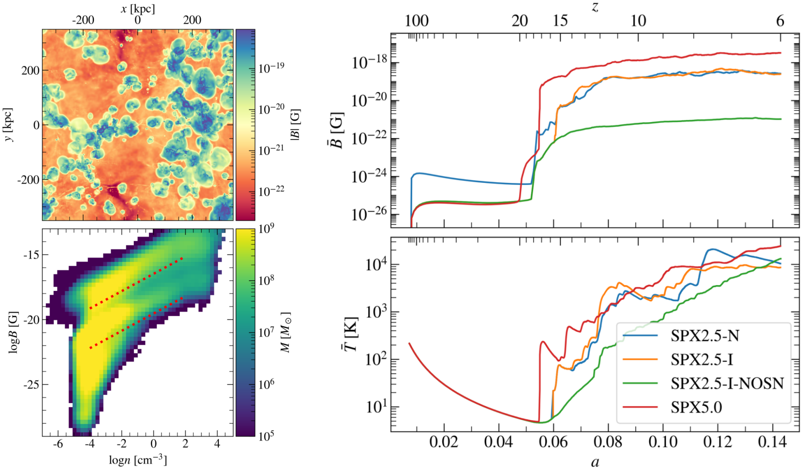
- Naively, going to smaller length scales L should increase B. However, in order to explain the *coherence on scales of several kpcs*, we would have to evoke a process such as a small-scale wind that moves the magnetic fields back to kpc scales and in that process we would have to account for adiabatic losses that accompany the expansion from small to large scales: in the end we would gain nothing from running a Biermann battery on smaller scales.
- This solves the cosmological magneto-genesis problem, but the big challenge remains in growing coherent large-scale magnetic fields from a small-amplitude, small-scale fields: this is a major challenge of dynamo theory!



- Cosmological simulations of the Biermann battery during the epoch of reionization with a state-of-the-art galaxy formation model find magnetic field generation at reionization fronts and at supernova-driven outflows (Attia+ 2021)

The induction equation – 1

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  1. an evolution equation of the magnetic field embedded in a fluid flow, and
  2. work out the magnetic force and stress.

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AIP

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- If the fluid moves with the velocity $\boldsymbol{v}$ relative to the observer, we have the component of the electric field parallel to the flow, $E_{\parallel} = \boldsymbol{E} \cdot \boldsymbol{v}/v$, and perpendicular to the flow, $\boldsymbol{E}_{\perp} = \boldsymbol{E} - E_{\parallel} \boldsymbol{v}/v$. The Lorentz transformation into observer's frame (unprimed quantities) is given by

$$E'_{\parallel} = E_{\parallel},$$

$$\boldsymbol{E}'_{\perp} = \gamma \left(\boldsymbol{E}_{\perp} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B} \right),$$

  where $\gamma = (1 - v^2/c^2)^{-1/2}$ is the Lorentz factor.

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  where $\gamma = (1 - v^2/c^2)^{-1/2}$ is the Lorentz factor.
- In the non-relativistic limit ($\boldsymbol{j}' = \boldsymbol{j}$ and $\gamma = 1$), we have Ohm's law in the observer's frame:

$$\boldsymbol{E} = -\frac{\boldsymbol{v}}{c} \times \boldsymbol{B} + \eta\, \boldsymbol{j}.$$

AIP

The induction equation – 2

- Let's recall Ohm's law:

$$\boldsymbol{E} = -\frac{\boldsymbol{v}}{c} \times \boldsymbol{B} + \eta\,\boldsymbol{j},$$

- Using Faraday's Law, $\dfrac{\partial \boldsymbol{B}}{\partial t} = -c\,\boldsymbol{\nabla} \times \boldsymbol{E}$, we get

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B}) - \boldsymbol{\nabla} \times (c\,\eta\,\boldsymbol{j}).$$

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- Using Ampère's law at small frequencies, $\boldsymbol{\nabla} \times \boldsymbol{B} = 4\pi \boldsymbol{j}/c$, we get

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B}) - \frac{c^2}{4\pi} \boldsymbol{\nabla} \times (\eta \boldsymbol{\nabla} \times \boldsymbol{B}).$$

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- Assuming $\eta = \text{const.}$, using the identity $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{B}) \equiv \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{B}) - \boldsymbol{\nabla}^2\boldsymbol{B}$ and the solenoidal condition, $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$, we arrive at the *induction equation*:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B}) + D\boldsymbol{\nabla}^2\boldsymbol{B}, \quad \text{where} \quad D = \frac{c^2\,\eta}{4\pi}.$$

AIP

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- At small frequencies (when the displacement current in Ampère's law is negligible), the induction equation has changed character: from Faraday's law describing the generation of voltages by changing magnetic fields in coils, we found an evolution equation of the magnetic field embedded in a fluid flow.

The induction equation – discussion

- The magnetic induction equation reads:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B}) + D\boldsymbol{\nabla}^2\boldsymbol{B}, \quad \text{where} \quad D = \frac{c^2\,\eta}{4\pi}. \tag{2}$$

- 1$^{\text{st}}$ term: the “convective term” states that the field is frozen into the flow (as we will see momentarily): an important property for astrophysical plasmas!
- 2$^{\text{nd}}$ term: the “diffusive term” represents the diffusive leakage of magnetic field lines across the conducting field, which is important for changing the magnetic topology, e.g. in reconnection.

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- 2^{nd} term: the "diffusive term" represents the diffusive leakage of magnetic field lines across the conducting field, which is important for changing the magnetic topology, e.g. in reconnection.
- The "diffusive term" can be neglected for infinite conductivity $\sigma = \eta^{-1}$ or for large magnetic Reynolds numbers $\mathrm{Re_m} \to \infty$:

$$\mathrm{Re_m} = \frac{|\text{convective term}|}{|\text{diffusive term}|} = \frac{L^{-1}\,v\,B}{D\,L^{-2}B} = \frac{L\,v}{D}$$

Magnetic force and stress – 1

Using Ampère's law at low frequencies, $\boldsymbol{\nabla} \times \boldsymbol{B} = 4\pi \boldsymbol{j}/c$, we will now show that the Lorentz force density can be written as follows:

$$\boldsymbol{f}_\mathrm{L} = \frac{1}{c}\boldsymbol{j} \times \boldsymbol{B} = \frac{1}{4\pi}(\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B} = \frac{1}{4\pi}(\boldsymbol{B} \cdot \boldsymbol{\nabla})\boldsymbol{B} - \frac{1}{8\pi}\boldsymbol{\nabla} B^2 = -\boldsymbol{\nabla} \cdot \mathbf{M},$$

where

$$\mathrm{M}_{ij} = -\frac{1}{4\pi}B_i B_j + \frac{1}{8\pi}B^2\delta_{ij}$$

is the magnetic stress tensor: it plays a role analogous to the fluid pressure in ordinary fluid mechanics (explaining the minus sign introduced in its definition).

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- We have

$$\begin{aligned}\left.(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B}\right|_i &= \varepsilon_{ijk}\varepsilon_{jlm}(\partial_l B_m)B_k=\varepsilon_{kij}\varepsilon_{jlm}(\partial_l B_m)B_k\\ &=(\delta_{kl}\delta_{im}-\delta_{km}\delta_{il})(\partial_l B_m)B_k=(\partial_k B_i)B_k-(\partial_i B_k)B_k\\ &=\left[(\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{B}-\frac{1}{2}\boldsymbol{\nabla}\boldsymbol{B}^2\right]_i.\end{aligned}$$

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- In this notation, the magnetic *curvature force* and the *magnetic pressure* force are not separated.

Magnetic force and stress – 2

- In order to fully separate the effects of magnetic curvature and pressure we write $\boldsymbol{B} = B\boldsymbol{b}$, where $\boldsymbol{b}$ is the unit vector in the direction of $\boldsymbol{B}$ and obtain

$$
\begin{aligned}
\boldsymbol{f}_\mathrm{L} &= \frac{1}{4\pi}\left(\boldsymbol{B}\cdot\boldsymbol{\nabla}\right)\boldsymbol{B} - \frac{1}{8\pi}\boldsymbol{\nabla}B^2 \\
&= \frac{B^2}{4\pi}\left(\boldsymbol{b}\cdot\boldsymbol{\nabla}\right)\boldsymbol{b} + \frac{1}{8\pi}\boldsymbol{b}(\boldsymbol{b}\cdot\boldsymbol{\nabla})B^2 - \frac{1}{8\pi}\boldsymbol{\nabla}B^2 \\
&= \frac{B^2}{4\pi}\left(\boldsymbol{b}\cdot\boldsymbol{\nabla}\right)\boldsymbol{b} - \frac{1}{8\pi}\boldsymbol{\nabla}_\perp B^2 \equiv \boldsymbol{f}_\mathrm{c} + \boldsymbol{f}_\mathrm{p},
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  where we define the gradient perpendicular to the magnetic field lines, $\boldsymbol{\nabla}_\perp = (\boldsymbol{1} - \boldsymbol{bb})\cdot\boldsymbol{\nabla}$.
- The second term, $\boldsymbol{f}_\mathrm{p}$, acts like a *pressure force perpendicular to the magnetic field lines* and the first term, $\boldsymbol{f}_\mathrm{c}$, is the *magnetic curvature force* that also acts in a plane orthogonal to the field line.

AIP

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  where we define the gradient perpendicular to the magnetic field lines, $\boldsymbol{\nabla}_{\perp} = (\mathbf{1} - \boldsymbol{bb})\cdot\boldsymbol{\nabla}$.
- The second term, $\boldsymbol{f}_{\mathrm{p}}$, acts like a *pressure force perpendicular to the magnetic field lines* and the first term, $\boldsymbol{f}_{\mathrm{c}}$, is the *magnetic curvature force* that also acts in a plane orthogonal to the field line.
- To see this, we locally identify a curved field line with its curvature circle so that we can locally define an azimuthally directed field $\boldsymbol{B} = B\boldsymbol{e}_{\varphi}$ in cylindrical coordinates (R, φ, z). Hence, in this case we obtain

$$(\boldsymbol{b}\cdot\boldsymbol{\nabla})\boldsymbol{b} = (\boldsymbol{e}_{\varphi}\cdot\boldsymbol{\nabla})\boldsymbol{e}_{\varphi} = -\frac{\boldsymbol{e}_R}{R}$$

  so that the curvature force always points towards the center of the curvature circle and aims to reduce the curvature by pulling the field line straight with a force that is the greater the smaller the curvature radius is.

Magnetic force and stress – 3

To get a better understanding, we show that the surface force (per unit area) exerted *by* a bounded volume V *on* its surroundings is given by

$$\boldsymbol{f}_S = \boldsymbol{n} \cdot \mathbf{M} = -\frac{1}{4\pi}\boldsymbol{B}B_n + \frac{1}{8\pi}B^2\boldsymbol{n},$$

where $B_n = \boldsymbol{B} \cdot \boldsymbol{n}$ is the component of $\boldsymbol{B}$ along the outward normal $\boldsymbol{n}$ to the surface of the volume.

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- The net Lorentz force acting *on* a volume V of fluid can be written as an integral of a magnetic stress vector acting on its surface,

$$\int_V \boldsymbol{f}_\mathrm{L} \mathrm{d}V = \int_V \frac{1}{4\pi}(\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B}\,\mathrm{d}V = -\int_V \boldsymbol{\nabla} \cdot \mathbf{M}\,\mathrm{d}V = -\oint_S \boldsymbol{n} \cdot \mathbf{M}\,\mathrm{d}S.$$

Magnetic force and stress – 3

To get a better understanding, we show that the surface force (per unit area) exerted *by* a bounded volume V *on* its surroundings is given by

$$\boldsymbol{f}_S = \boldsymbol{n} \cdot \mathbf{M} = -\frac{1}{4\pi} \boldsymbol{B} B_n + \frac{1}{8\pi} B^2 \boldsymbol{n},$$

where $B_n = \boldsymbol{B} \cdot \boldsymbol{n}$ is the component of $\boldsymbol{B}$ along the outward normal $\boldsymbol{n}$ to the surface of the volume.

- The net Lorentz force acting *on* a volume V of fluid can be written as an integral of a magnetic stress vector acting on its surface,

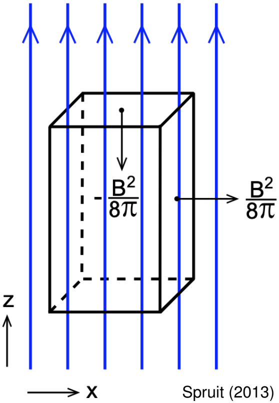
$$\int_V \boldsymbol{f}_{\mathrm{L}} \mathrm{d}V = \int_V \frac{1}{4\pi} (\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B} \, \mathrm{d}V = -\int_V \boldsymbol{\nabla} \cdot \mathbf{M} \, \mathrm{d}V = -\oint_S \boldsymbol{n} \cdot \mathbf{M} \, \mathrm{d}S.$$

- To get the force $\boldsymbol{f}_S$ exerted *by* the volume *on* its surroundings, we need to add a minus sign to the last term,

$$\boldsymbol{f}_S = \boldsymbol{n} \cdot \mathbf{M} = -\frac{1}{4\pi} \boldsymbol{B} B_n + \frac{1}{8\pi} B^2 \boldsymbol{n},$$

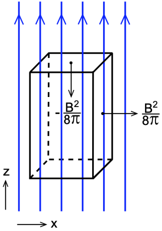
  where $B_n = \boldsymbol{B} \cdot \boldsymbol{n}$.

Magnetic force and stress – 4



Spruit (2013)

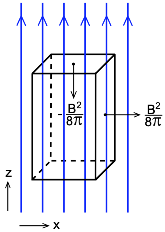
- We take a uniform magnetic field ($\boldsymbol{B} = B\boldsymbol{e}_z$) and compute the surface forces $\boldsymbol{f}_S$ exerted by a rectangular volume that is aligned with the magnetic field.
- Symmetry limits the surface forces to two different types: 4 with a normal perpendicular to $\boldsymbol{B}$ and 2 with a normal that is (anti-)parallel to $\boldsymbol{B}$.
- Which magnetic forces (pressure or tension) contribute to these surface forces?



- Take the *surface* perpendicular to the x axis on the right-hand side of the box:

$$\boldsymbol{n} = \boldsymbol{e}_x : \qquad \boldsymbol{f}_{\text{right}} = \boldsymbol{e}_x \cdot \mathbf{M},$$

$$f_{\text{right},\,x} = -\frac{1}{4\pi} B_x B_z + \frac{1}{8\pi} B^2 = \frac{1}{8\pi} B^2, \quad f_{\text{right},\,y} = f_{\text{right},\,z} = 0.$$



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- The stress exerted by the magnetic field at the top of the surface element is

$$\boldsymbol{n} = \boldsymbol{e}_z : \qquad \boldsymbol{f}_{\text{top}} = \boldsymbol{e}_z \cdot \mathbf{M}, \qquad \text{note that we have here: } B^2 = B_z^2,$$

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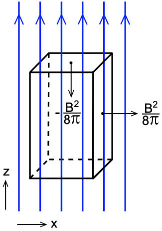
Magnetic force and stress – 5



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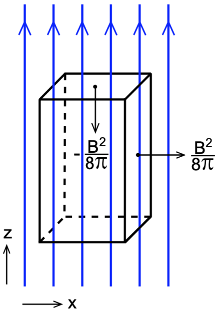
The stress is also perpendicular to the surface and of equal magnitude to that of the magnetic pressure exerted at the vertical surfaces, but of opposite sign!

Magnetic force and stress – 6

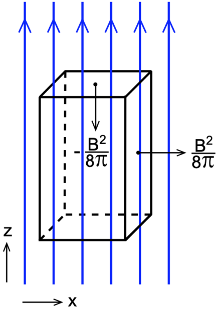


Conclusions:

- The magnetic pressure causes the fluid volume to expand in the perpendicular directions to the magnetic field (in x and y for a field in z direction)

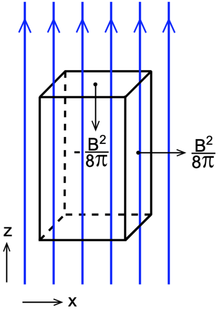
AIP

Magnetic force and stress – 6



Conclusions:

- The magnetic pressure causes the fluid volume to expand in the perpendicular directions to the magnetic field (in x and y for a field in z direction)
- Because there are no magnetic monopoles (i.e., $\nabla \cdot \boldsymbol{B} = 0$), magnetic field lines have no 'ends'. Hence, the contraction along the magnetic field under magnetic stress does not happen in practice because the tension at its top and bottom surfaces is exactly balanced by the tension in the magnetic lines continuing above and below the box.



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- The effects of tension in a magnetic field manifest themselves through the curvature of field lines.

Magneto-hydrodynamics

- For a collisional fluid on scales larger than the particle mean-free path and on time scales longer than the inverse plasma frequency, $\tau > \omega_{\mathrm{pl}}^{-1}$, the evolution of the magnetic vector field $\boldsymbol{B}$ is given by magneto-hydrodynamics (MHD).
- Ideal MHD assumes an inviscid (i.e., no viscosity), ideally conducting fluid.

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- Ideal MHD assumes an inviscid (i.e., no viscosity), ideally conducting fluid.
- To derive MHD, we add the Lorentz force to the momentum evolution equation (the Euler equation) and supplement the system of conservation equations of mass, momentum and entropy by the equation for magnetic induction, Eq. (2) without the diffusion term and obtain the equations of ideal MHD:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = 0,$$

$$\rho \left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} \right) = -\boldsymbol{\nabla} P + \boldsymbol{j} \times \boldsymbol{B} = -\boldsymbol{\nabla} \cdot \left[\left(P + \frac{\boldsymbol{B}^2}{8\pi} \right) \mathbf{1} - \frac{1}{4\pi} \boldsymbol{B}\boldsymbol{B}^{\mathrm{T}} \right],$$

$$\frac{\partial s}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} s = 0,$$

$$\frac{\partial \boldsymbol{B}}{\partial t} - \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B}) = \mathbf{0}, \quad \text{subject to the constraint} \quad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0,$$

where $\rho = \rho(\boldsymbol{x}, t)$, $P = P(\boldsymbol{x}, t)$, $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x}, t)$, $\boldsymbol{j} = \boldsymbol{j}(\boldsymbol{x}, t)$, $s = s(\boldsymbol{x}, t)$, and $\boldsymbol{B} = \boldsymbol{B}(\boldsymbol{x}, t)$ are the density, pressure, velocity, electric current, entropy, and magnetic field.

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Magnetic flux freezing – 1

- To show that the magnetic flux is "frozen" into the plasma, we start with the induction equation (2) without the diffusion term:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B}),$$

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$$\frac{\partial \boldsymbol{B}}{\partial t} = (\boldsymbol{B} \cdot \boldsymbol{\nabla})\boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{B} - (\boldsymbol{\nabla} \cdot \boldsymbol{v})\boldsymbol{B},$$

  which can be rearranged to yield

$$\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t} \equiv \frac{\partial \boldsymbol{B}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{B} = (\boldsymbol{B} \cdot \boldsymbol{\nabla})\boldsymbol{v} - (\boldsymbol{\nabla} \cdot \boldsymbol{v})\boldsymbol{B}.$$

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- Multiplying this equation by ρ^{-1} and rearranging terms yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{B}}{\rho}\right) = \frac{1}{\rho}\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t} - \frac{\boldsymbol{B}}{\rho^2}\frac{\mathrm{d}\rho}{\mathrm{d}t} = \left(\frac{\boldsymbol{B}}{\rho} \cdot \boldsymbol{\nabla}\right)\boldsymbol{v},$$

  This is the flux-freezing equation of magnetic fields.

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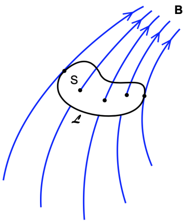
Magnetic flux freezing – 2



Spruit (2013)

Flux freezing condition: $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{B}}{\rho}\right) = \left(\frac{\boldsymbol{B}}{\rho}\cdot\boldsymbol{\nabla}\right)\boldsymbol{v}$

- Consider the evolution of $\delta\boldsymbol{x}$ which connects two neighboring points in the fluid:

$$\begin{aligned}
\Delta\boldsymbol{x}(t) &= \delta\boldsymbol{x} \\
\Delta\boldsymbol{x}(t+\Delta t) &= \delta\boldsymbol{x} + (\delta\boldsymbol{x}\cdot\boldsymbol{\nabla})\,\boldsymbol{v}\,\Delta t + \mathcal{O}(\Delta t^2) \\
\frac{\mathrm{d}\delta\boldsymbol{x}}{\mathrm{d}t} &= \frac{\Delta\boldsymbol{x}(t+\Delta t) - \Delta\boldsymbol{x}(t)}{\Delta t} = (\delta\boldsymbol{x}\cdot\boldsymbol{\nabla})\,\boldsymbol{v}
\end{aligned}$$

- $\boldsymbol{B}/\rho$ and $\delta\boldsymbol{x}$ satisfy the same *ordinary differential equation*, hence if initially $\delta\boldsymbol{x} = \varepsilon\boldsymbol{B}/\rho$, the same relation will hold for all times. If $\delta\boldsymbol{x}$ connects two particles on the same field line then they remain on the same field line.

Magnetic flux freezing – 3

Flux freezing condition: $$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{B}}{\rho}\right) = \left(\frac{\boldsymbol{B}}{\rho}\cdot\boldsymbol{\nabla}\right)\boldsymbol{v}$$

- What does this flux-freezing condition imply for a uniform contraction/expansion of the plasma?

AIP

Magnetic flux freezing – 3

Flux freezing condition: $$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{B}}{\rho}\right) = \left(\frac{\boldsymbol{B}}{\rho}\cdot\boldsymbol{\nabla}\right)\boldsymbol{v}$$

- What does this flux-freezing condition imply for a uniform contraction/expansion of the plasma?
- The plasma resides in a sphere of radius r and conserves mass and magnetic flux $\mathrm{d}\Phi = \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{A}$ (where $\mathrm{d}\boldsymbol{A}$ is the surface element on the sphere). Thus, both ρr^3 and Br^2 are constant and we obtain

$$B \equiv \sqrt{\langle \boldsymbol{B} \rangle} \propto r^{-2} \propto \rho^{\alpha_B}, \quad \alpha_B = \frac{2}{3},$$

for isotropic contraction or expansion of either a homogeneous, fully turbulent or radial field.

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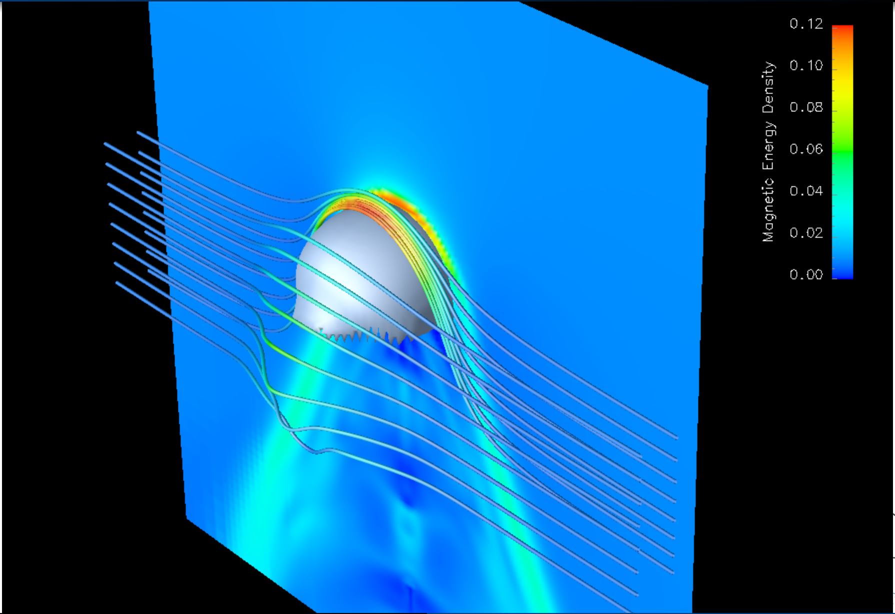
  for isotropic contraction or expansion of either a homogeneous, fully turbulent or radial field.
- Note that the scaling exponent α_B depends on the type of symmetry invoked during collapse (whether it is isotropic or not) and can differ for contraction along a homogeneous magnetic field ($\alpha_B = 0$) or perpendicular to it ($\alpha_B = 1$).
- Thus, flux freezing alone predicts a tight relation between B and ρ. Moreover, it has a surprising property called magnetic draping.

What is magnetic draping?

Interaction of an obstacle (Earth, star, galaxy, . . .) with a magnetized plasma



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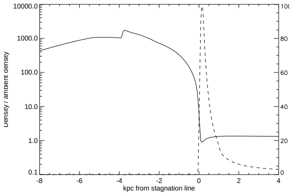
- Is magnetic draping similar to ram pressure compression?
  $\rightarrow$ no, the density is not increased in magnetic draping as shown by ideal MHD simulations *(right)*



Dursi & CP (2008)

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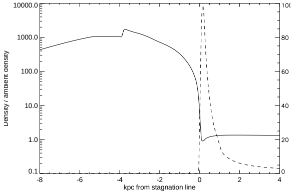
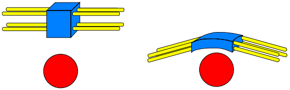
- Is magnetic flux still frozen into the plasma?
  yes, but plasma can also move along field lines while field lines get stuck at obstacle





Dursi & CP (2008)

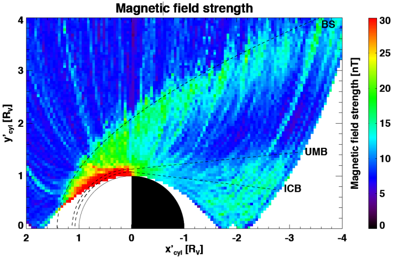
Applications of magnetic draping

- Solar-wind magnetic field is draped around the magnetopause of Earth: this protects Earth from cosmic rays during times of spin flip of the magnetic poles

- draping of solar-wind magnetic field at other moons and planets of the solar system: plasma physics

- hydrodynamic stability of underdense radio bubbles

- sharpness (T_e, n_e) of cold fronts: without B, smoothed out by diffusion and heat conduction on $\gtrsim 10^8$ yrs



Guicking+ (2010): magnetic draping around Venus

- magnetic draping on spiral galaxies in galaxy clusters: method for detecting the orientation of cluster magnetic fields

Dispersion relation and hydrodynamic waves – 1

- The hydrodynamic equations are (without viscosity and heat conduction):

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = 0,$$

$$\frac{\partial}{\partial t}(\rho \boldsymbol{v}) + \boldsymbol{\nabla} \cdot \left(\rho \boldsymbol{v} \boldsymbol{v}^{\mathrm{T}} + P\mathbf{1}\right) = 0,$$

$$\frac{\partial s}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} s = 0.$$

where ρ is mass density, $\boldsymbol{v}$ is mean velocity, and pressure P and entropy s are related by the equation of state $P(\rho, s)$.

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- **Dispersion relation for sound waves:** we perturb continuity and momentum equation, split the dynamical quantities into background values (that do not depend on time) and small perturbations: $\rho = \rho_0 + \delta\rho$, $\boldsymbol{v} = \delta\boldsymbol{v}$ (note: $\boldsymbol{v}_0 = \mathbf{0}$), $P = P_0 + \delta P$. The constraint equation for the background reads

$$\boldsymbol{\nabla} P_0 = \mathbf{0}. \tag{3}$$

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- The perturbed mass and momentum equations are to first order (using Eq. 3):

$$\begin{aligned}\frac{\partial \delta\rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho_0 \delta\boldsymbol{v} + \cancel{\delta\rho\boldsymbol{v}_0}) &= 0,\\ \frac{\partial}{\partial t}(\rho_0 \delta\boldsymbol{v}) + \boldsymbol{\nabla} \cdot (\cancel{\delta\rho\boldsymbol{v}_0\boldsymbol{v}_0^{\mathrm{T}}} + \cancel{\rho_0\delta\boldsymbol{v}\boldsymbol{v}_0^{\mathrm{T}}} + \mathbf{1}\delta P) &= \mathbf{0}.\end{aligned}$$

Dispersion relation and hydrodynamic waves – 2

- We recap the perturbed mass and momentum equations to first order:

$$\begin{aligned} \frac{\partial \delta\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho_0 \delta \boldsymbol{v}) &= 0 \qquad \Big| \partial_t(\,\cdot\,) \\ \frac{\partial}{\partial t}(\rho_0 \delta \boldsymbol{v}) + \boldsymbol{\nabla}\cdot(\mathbf{1}\delta P) &= \mathbf{0} \qquad \Big| \boldsymbol{\nabla}\cdot(\,\cdot\,) \end{aligned}$$

- Subtracting the second from the first equation yields a wave equation,

$$\partial_t^2 \delta\rho - \boldsymbol{\nabla}^2 \delta P = 0. \tag{4}$$

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- We Fourier transform the perturbations $\delta\rho$ and δP using the convention:

$$\delta\rho(\boldsymbol{x}, t) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3}\, \delta\hat{\rho}(\boldsymbol{k}, \omega)\, \mathrm{e}^{-\mathrm{i}\omega t + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}},$$

and decompose Eq. (4) into plane waves to obtain the dispersion relation:

$$-\omega^2 \delta\hat{\rho} + k^2 \delta\hat{P} = 0,$$

$$\omega^2 = \frac{\delta\hat{P}}{\delta\hat{\rho}} k^2 \qquad \Longrightarrow \qquad \omega = \pm\sqrt{\frac{\delta\hat{P}}{\delta\hat{\rho}}}\, k,$$

Dispersion relation and hydrodynamic waves – 3

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How many equations do you have and how many eigenvalues does the linearized system of equations allow for?

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  How many equations do you have and how many eigenvalues does the linearized system of equations allow for?
- By combining the first four equations (for mass and momentum density) and adopting the equation of state $P = P(\rho, s)$, we get

$$\omega = \pm\sqrt{\frac{\delta\hat{P}}{\delta\hat{\rho}}}\,k = \pm\sqrt{\left.\frac{\partial P}{\partial \rho}\right|_s}\,k, \qquad \Longrightarrow \qquad c_{\mathrm{s}} = \frac{\omega}{k} = \frac{\partial \omega}{\partial k} = \sqrt{\left.\frac{\partial P}{\partial \rho}\right|_s}.$$

  i.e., the sound wave is a degenerate solution and accounts for four eigenvalues.

Dispersion relation and hydrodynamic waves – 3

- The hydrodynamic equations are (without viscosity and heat conduction):

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = 0,$$
$$\frac{\partial}{\partial t}(\rho \boldsymbol{v}) + \boldsymbol{\nabla} \cdot \left(\rho \boldsymbol{v}\boldsymbol{v}^{\mathrm{T}} + P\boldsymbol{1}\right) = 0,$$
$$\frac{\partial s}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} s = 0.$$

  How many equations do you have and how many eigenvalues does the linearized system of equations allow for?

- By combining the first four equations (for mass and momentum density) and adopting the equation of state $P = P(\rho, s)$, we get

$$\omega = \pm\sqrt{\frac{\delta \hat{P}}{\delta \hat{\rho}}}\, k = \pm\sqrt{\left.\frac{\partial P}{\partial \rho}\right|_s}\, k, \qquad \Longrightarrow \qquad c_{\mathrm{s}} = \frac{\omega}{k} = \frac{\partial \omega}{\partial k} = \sqrt{\left.\frac{\partial P}{\partial \rho}\right|_s}.$$

  i.e., the sound wave is a degenerate solution and accounts for four eigenvalues.

- Perturbing the entropy equation yields to first order in Fourier space

$$\mathrm{i}\omega\delta\hat{s} - \delta\hat{\boldsymbol{v}} \cdot \boldsymbol{\nabla} s_0 = 0$$
$$\Longrightarrow \quad \omega = 0 \quad \text{and} \quad s_0 = \text{const.}$$

  The entropy mode is a compressible zero-frequency mode with eigenfunctions $\delta P = \delta \boldsymbol{v} = \delta \boldsymbol{B} = 0$ and $\delta T/T = -\delta\rho/\rho = 2\delta s/5$.

AIP

Magneto-hydrodynamic waves – 1

- Add magnetic fields to the system in the ideal MHD approximation. How many equations and eigenvalues do you have now?

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$$\rho \left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} \right) = -\boldsymbol{\nabla} P + \boldsymbol{j} \times \boldsymbol{B} = -\boldsymbol{\nabla} \cdot \left[\left(P + \frac{\boldsymbol{B}^2}{8\pi} \right) \mathbf{1} + \frac{1}{4\pi} \boldsymbol{B}\boldsymbol{B}^{\mathrm{T}} \right],$$

$$\frac{\partial s}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} s = 0,$$

$$\frac{\partial \boldsymbol{B}}{\partial t} - \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B}) = \mathbf{0}, \quad \text{subject to the constraint} \quad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0,$$

where $\rho = \rho(\boldsymbol{x})$, $P = P(\boldsymbol{x})$, $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x})$, $\boldsymbol{j} = \boldsymbol{j}(\boldsymbol{x})$, $s = s(\boldsymbol{x})$, and $\boldsymbol{B} = \boldsymbol{B}(\boldsymbol{x})$ are the density, pressure, velocity, electric current, entropy, and magnetic field.

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- There are a total of 8 equations: 5 hydrodynamics equations plus 3 components of the induction equation. However, the constraint equation, $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$, reduces the dimensionality to seven degrees of freedom.

Magneto-hydrodynamic waves – 2

In a magnetized plasma, there are seven different wave modes:

- 2 shear Alfvén waves: ***incompressible*** **(**$\delta\rho = 0$**)** ***and transverse polarized waves;*** restoring force provided by magnetic tension; propagate obliquely and parallel to $\boldsymbol{B}$ with $v_\mathrm{a} = B/\sqrt{4\pi\,\rho}$
- 2 fast magnetosonic waves: ***compressible, longitudinal waves;*** restoring force provided by (thermal and magnetic) pressure; propagate parallel and perpendicular to $\boldsymbol{B}$; equivalent to sound waves in high-β plasmas, where $\beta = P_\mathrm{th}/P_B = 2c_\mathrm{s}/v_\mathrm{a}$
- 2 slow magnetosonic waves: ***compressible, longitudinal waves;*** restoring force provided by thermal pressure; propagate only parallel to $\boldsymbol{B}$; equivalent to compressible Alfvén waves in high-β plasma
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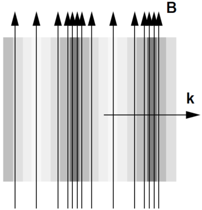
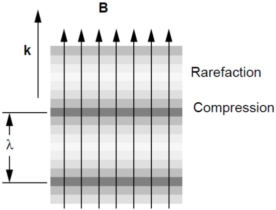
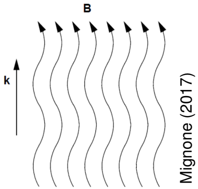


fast magnetosonic wave

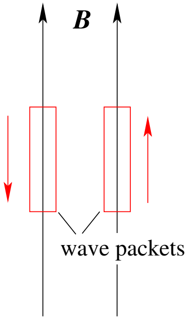


fast or slow sound wave



shear Alfvén wave

Alfvénic turbulence – the picture



Interacting Alfvén wave packets.

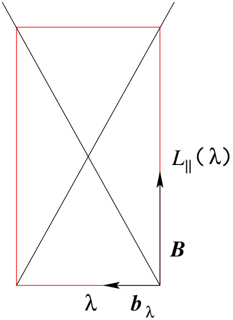
- Alfvénic turbulence is incompressible:

$$\frac{\delta v_a}{v_a} = \frac{\delta B}{B}$$

- What happens when the two wave packets are interacting?
- The down-going packet causes field line wandering such that the upward going packet is broken apart after a distance $L_{||}(\lambda)$.
- In other words, the travel time across this wave package in the direction of the mean magnetic field equals the eddy turn-over time in the perpendicular direction.
- This gives rise to the critical balance condition of Alfvénic turbulence

  (Goldreich & Shridhar 1995, 1997, Lithwick & Goldreich 2001)

Alfvénic turbulence - the scaling



Geometrical interpretation of the "critical balance" condition.

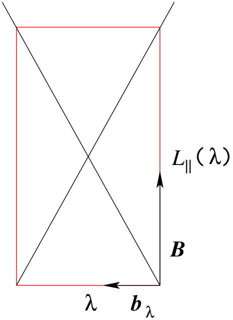
- The critical balance condition reads:

$$L_\parallel = \frac{\lambda B}{b_\lambda}$$

- In Kolmogorov turbulence, the energy flux of the fluctuating field at scale λ is constant, $b_\lambda^2/t_\lambda = \text{const.}$ Equating the wave travel time along $\boldsymbol{B}$, $t_\parallel$, with the eddy turn-over time in the perpendicular direction, t_λ, we get

$$t_\parallel = \frac{L_\parallel}{v_a} = \frac{\lambda\, B}{v_a\, b_\lambda} = t_\lambda \propto b_\lambda^2,$$

AIP



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$$t_{\|} = \frac{L_{\|}}{v_\text{a}} = \frac{\lambda\, B}{v_\text{a}\, b_\lambda} = t_\lambda \propto b_\lambda^2,$$

- Because $B \propto v_\text{a} = \text{const}$. in incompressible turbulence, we obtain the scaling of Alfvénic turbulence:

$$b_\lambda \propto \lambda^{1/3} \quad \text{or} \quad L_{\|} \propto \lambda^{2/3}\, L_\text{MHD}^{1/3}$$

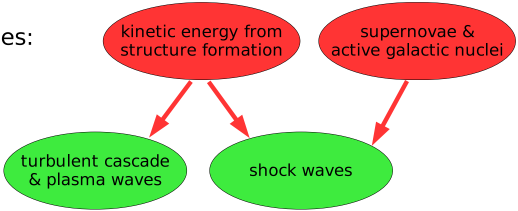
$\Rightarrow$ the smaller the scale λ, the more anisotropic is the turbulent scaling and the more elongated are the eddies ($L_{\|}/\lambda \propto \lambda^{-1/3}$) whose long axis is aligned with the local $\langle B \rangle$!

Multi messenger approach for non-thermal processes

Relativistic populations and radiative processes in clusters:

Energy sources:

Plasma processes:



CP+ (2008)

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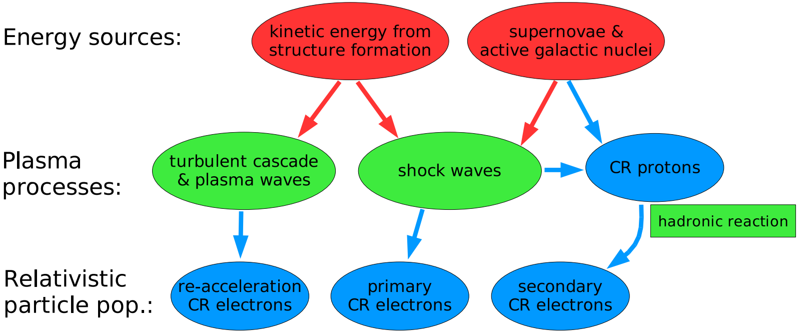

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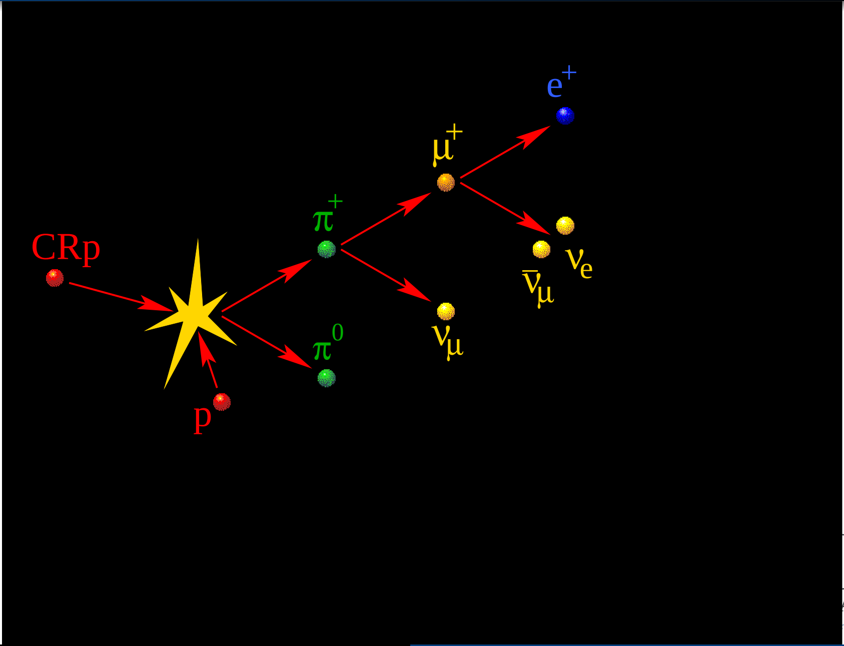
Hadronic cosmic ray proton interaction



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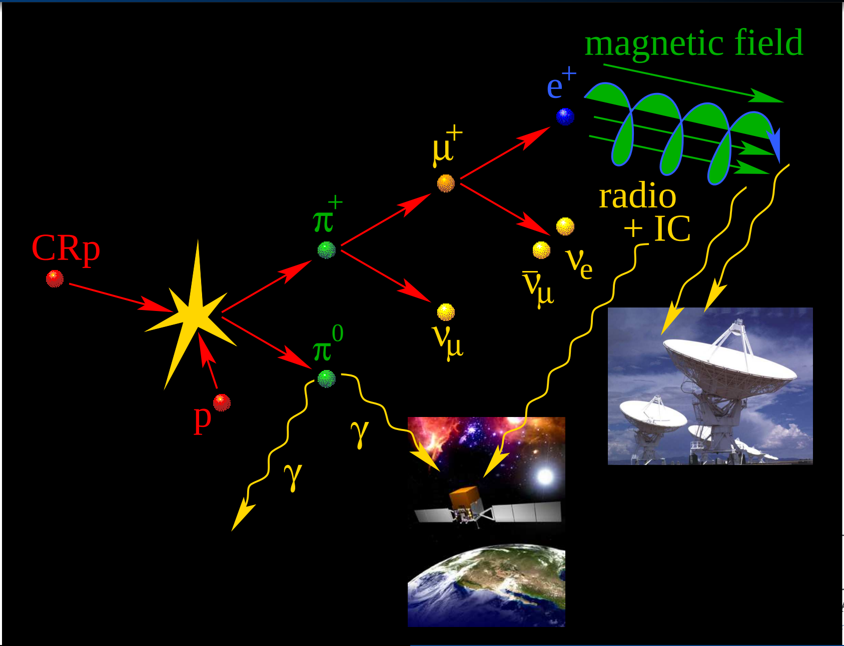

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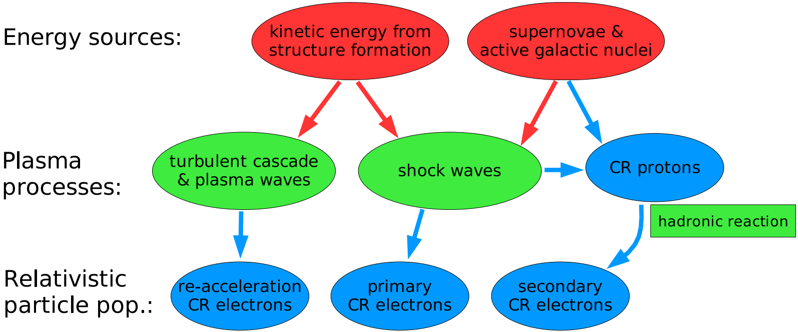

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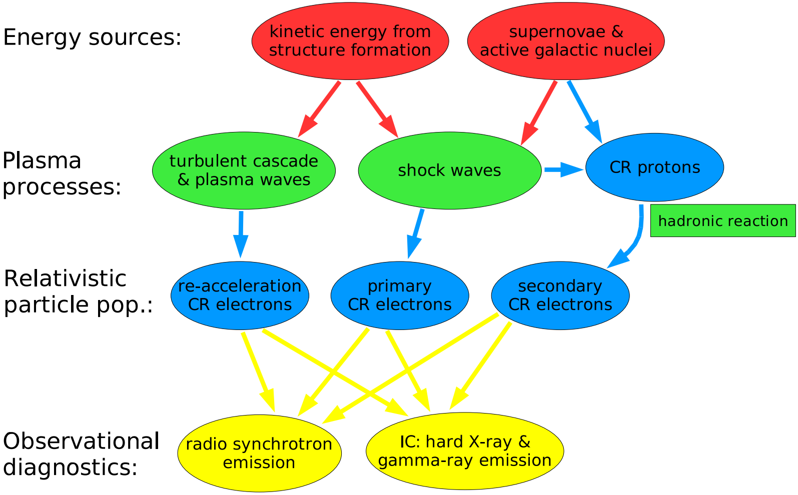

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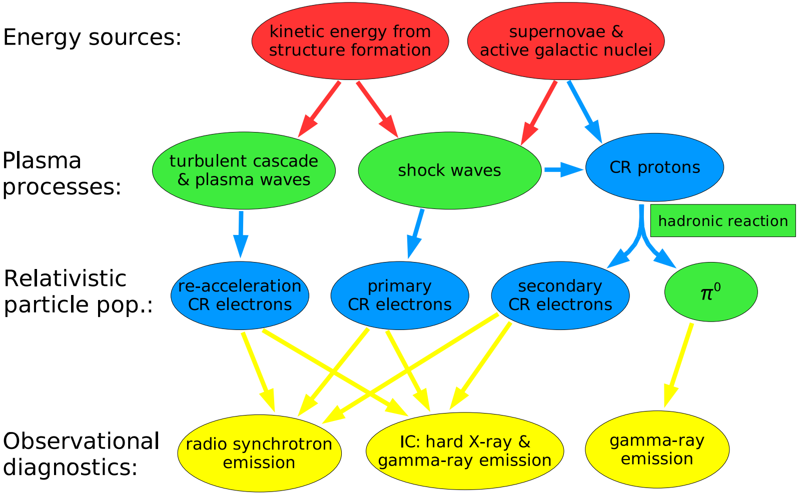

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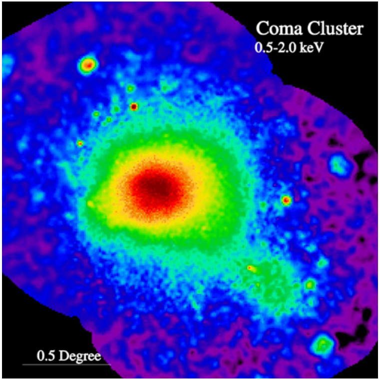
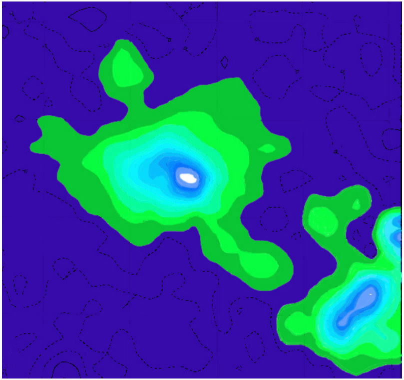


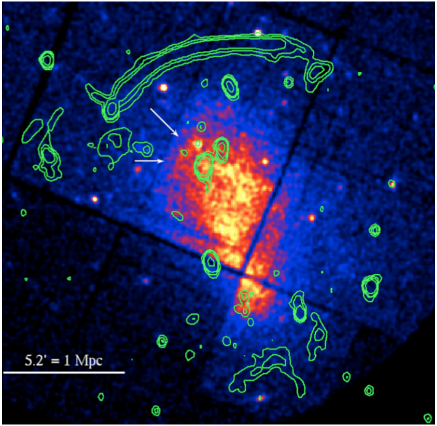
thermal X-ray emission

Snowden/MPE/ROSAT



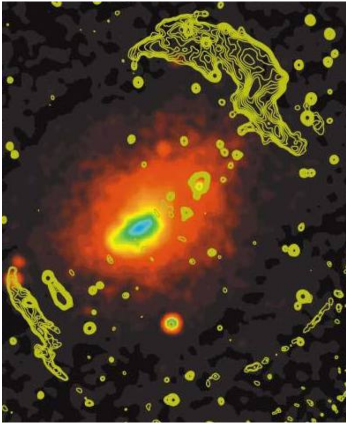
radio synchrotron emission

Deiss/Effelsberg



CIZA J2242.8+5301 ("sausage relic")

X-ray: XMM; radio: WSRT/Ogrean



Abell 3667

radio: Johnston-Hollitt; X-ray: ROSAT/PSPC

Cosmic rays and magnetic fields in clusters – 1

The energy loss rate of a relativistic electron of energy $E_\mathrm{e} = \gamma m_\mathrm{e} c^2$ is given by

$$\dot{E}_\mathrm{e} = \frac{\sigma_\mathrm{T} c}{6\pi} (B_\mathrm{cmb}^2 + B^2)\gamma^2,$$

where σ_T is the Thomson cross section, m_e is the electron rest mass, c is the light speed, γ is the Lorentz factor, B is the magnetic field strength and $B_\mathrm{cmb} \simeq 3.2\mu\mathrm{G}$ is the equivalent field of the cosmic microwave background energy density today.

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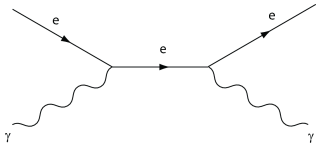
- first term $\propto B_{cmb}^2$ describes inverse Compton scattering off of cosmic microwave background photons, second term $\propto B^2$ describes synchrotron emission
- the same Feynman diagram describes both scattering processes with a real photon/virtual photon from the magnetic field



AIP

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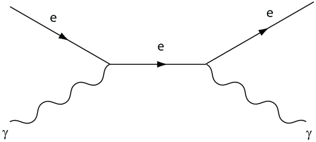

- the cooling time $t_\mathrm{cool} = E_\mathrm{e}/\dot{E}_\mathrm{e}$ of a relativistic electron is given by

$$t_\mathrm{cool} = \frac{E_\mathrm{e}}{\dot{E}_\mathrm{e}} = \frac{6\pi m_\mathrm{e} c}{\sigma_\mathrm{T} \left(B_\mathrm{cmb}^2 + B^2\right)\gamma} \approx 200\,\mathrm{Myr},$$

for $B = 1\ \mu$G and $\gamma = 10^4$.

Cosmic rays and magnetic fields in clusters – 2

- Recall the cooling time $t_{\rm cool} = E_{\rm e}/\dot{E}_{\rm e}$ of a relativistic electron,

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$$\nu_{\mathrm{synch}} = \frac{3eB}{2\pi\, m_{\mathrm{e}} c}\,\gamma^2 \simeq 1\ \mathrm{GHz}\,\frac{B}{\mu\mathrm{G}}\left(\frac{\gamma}{10^4}\right)^2.$$

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- Combining both equations by eliminating the Lorentz factor γ yields the cooling time of electrons that emit at frequency $\nu_{\rm syn}$,

$$t_{\rm cool} = \frac{\sqrt{54\pi m_{\rm e} c\, eB\nu_{\rm syn}^{-1}}}{\sigma_{\rm T}\,(B_{\rm cmb}^2 + B^2)} \lesssim 190\left(\frac{\nu_{\rm syn}}{1.4\,\mathrm{GHz}}\right)^{-1/2}\mathrm{Myr},$$

The cooling time $t_{\rm cool}$ is then bound from above and attains its maximum cooling time at $B = B_{{\rm cmb},0}/\sqrt{3} \simeq 1.8\,\mu\mathrm{G}$, independent of the magnetic field.

AIP

Giant radio relics in galaxy clusters

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- We assume that the relativistic electrons are accelerated at a strong cluster merger shock and are advected with the post-shock gas. Assuming that the incoming gas had a pre-shock velocity of $v_1 = 1200$ km/s in the shock frame, we get a post-shock velocity

$$v_2 = \frac{\rho_1}{\rho_2}\, v_1 = \frac{(\gamma - 1)\mathcal{M}_1^2 + 2}{(\gamma + 1)\mathcal{M}_1^2}\, v_1 = 400 \left(\frac{v_1}{1200\,{\rm km\,s^{-1}}}\right) {\rm km\,s^{-1}}$$

for a shock Mach number of $\mathcal{M}_1 = 3$.

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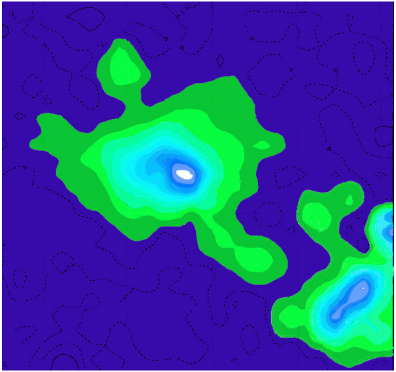
- This implies a maximum cooling length $L_{\rm cool,\,max} = v_2 t_{\rm cool,\,max} = 80$ kpc, which decreases for larger magnetic field strengths to assume a value of

$$L_{\rm cool} = v_2 t_{\rm cool} = \frac{v_2\sqrt{54\pi m_e c\, eB\nu_{\rm syn}^{-1}}}{\sigma_{\rm T}\,(B_{\rm cmb}^2 + B^2)} \approx 30 \left(\frac{\nu_{\rm syn}}{1.4\,{\rm GHz}}\right)^{-1/2} {\rm kpc}$$

  for 5 μG. Typical radial extends of radio relics are of that size. Hence, one can use the relic geometry to estimate magnetic field strengths (projection effects!).

Giant radio halos in galaxy clusters

- The maximum cooling length is $L_{\text{cool, max}} = v_2 t_{\text{cool, max}} = 80$ kpc at 1.4 GHz.
- The spatial extend of giant radio halos is ~ 2 Mpc and the emission is not polarized.



radio synchrotron emission (Deiss/Effelsberg)

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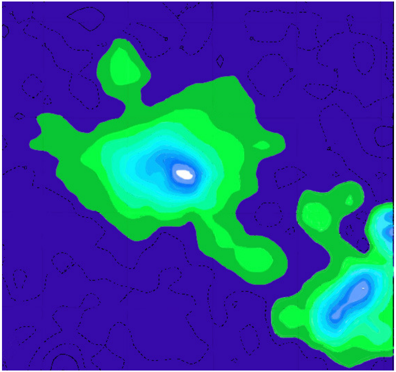
- Because synchrotron emission is intrinsically polarized, this means that the emission is a projection of causally uncorrelated regions along the line of sight or there is beam depolarization.



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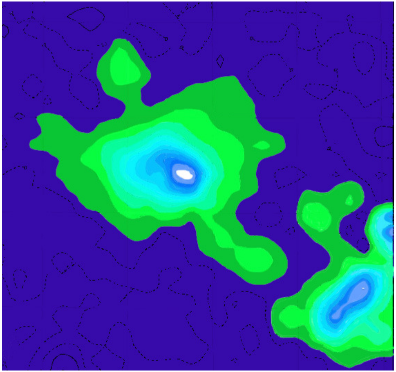

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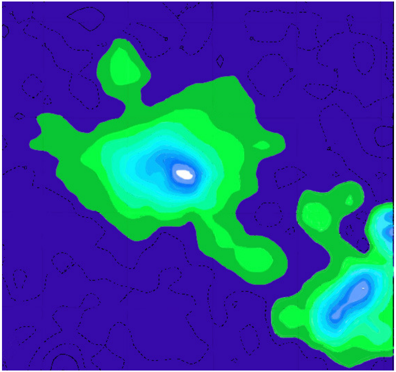

radio synchrotron emission (Deiss/Effelsberg)

- Hadronic model: relativistic protons interact hadronically with gas protons and produce secondary electrons/positrons that emit in the radio.

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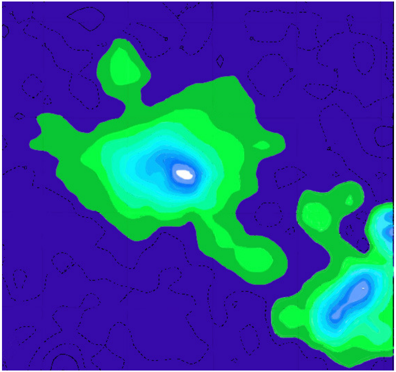
- Because synchrotron emission is intrinsically polarized, this means that the emission is a projection of causally uncorrelated regions along the line of sight or there is beam depolarization.
- Because $L_{\text{halo}} \approx 25 L_{\text{cool, max}}$ there must be a volume filling acceleration process of relativistic electrons.

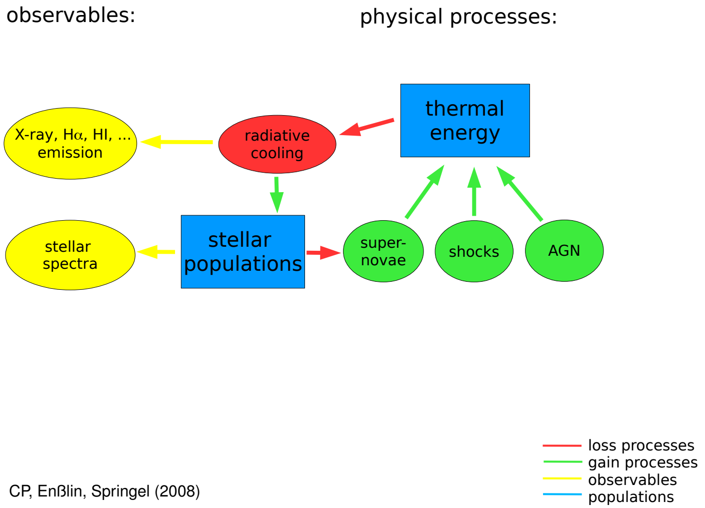


radio synchrotron emission (Deiss/Effelsberg)

- Hadronic model: relativistic protons interact hadronically with gas protons and produce secondary electrons/positrons that emit in the radio.
- Reacceleration model: fossil or secondary electrons interact with turbulent magneto-hydrodynamic waves and experience 2$^{\text{nd}}$ order Fermi acceleration that makes them visible at radio wave lengths.

AIP

Simulations – flowchart

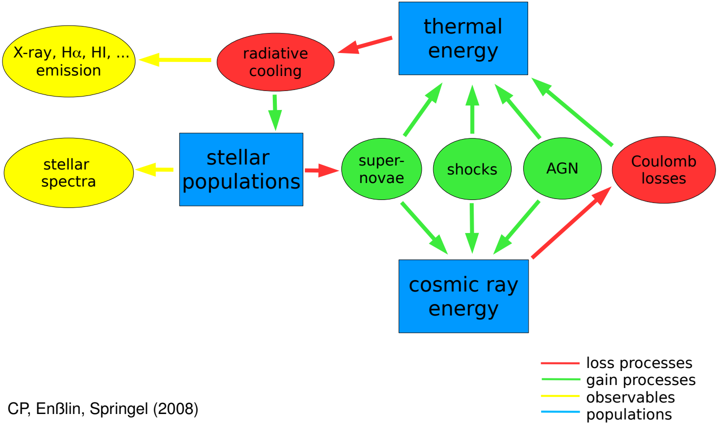


CP, Enßlin, Springel (2008)

Simulations with cosmic ray physics

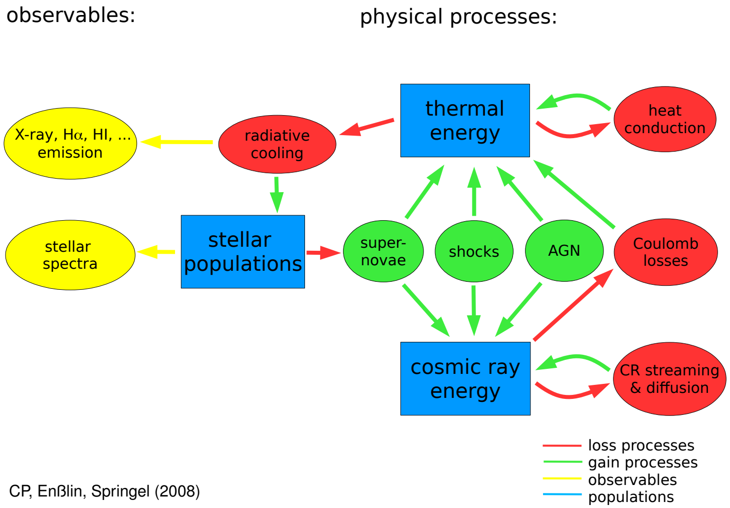
observables:

physical processes:



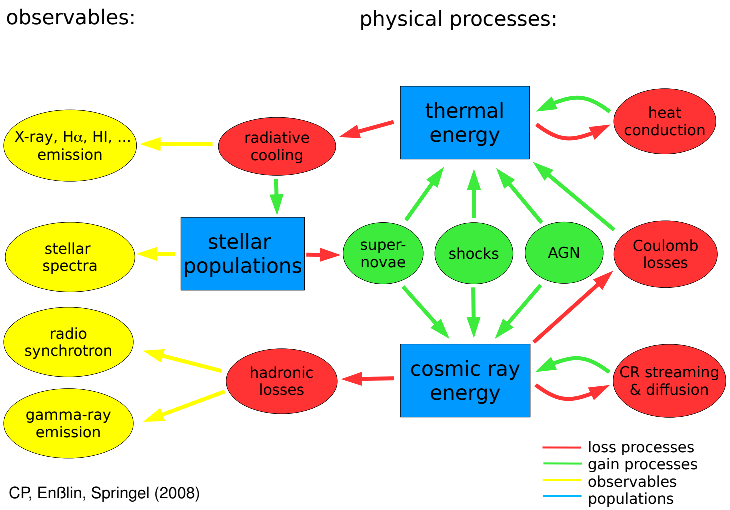
CP, Enßlin, Springel (2008)

Simulations with cosmic ray physics



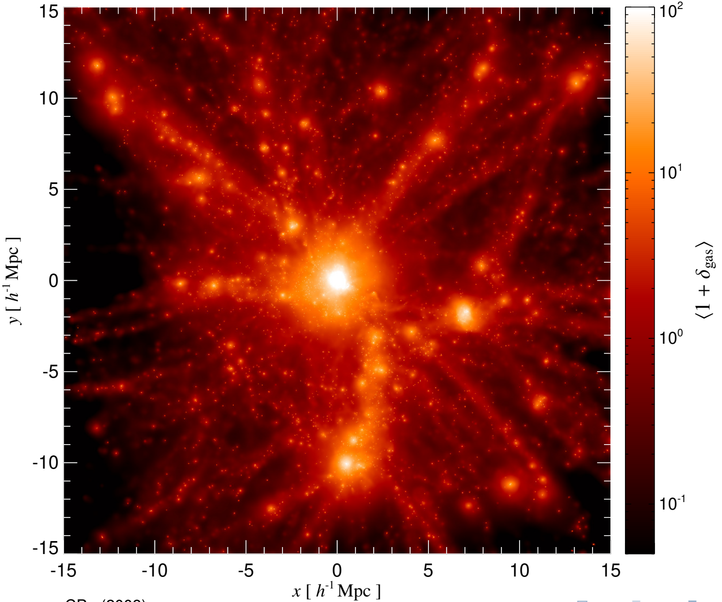
CP, Enßlin, Springel (2008)

Simulations with cosmic ray physics



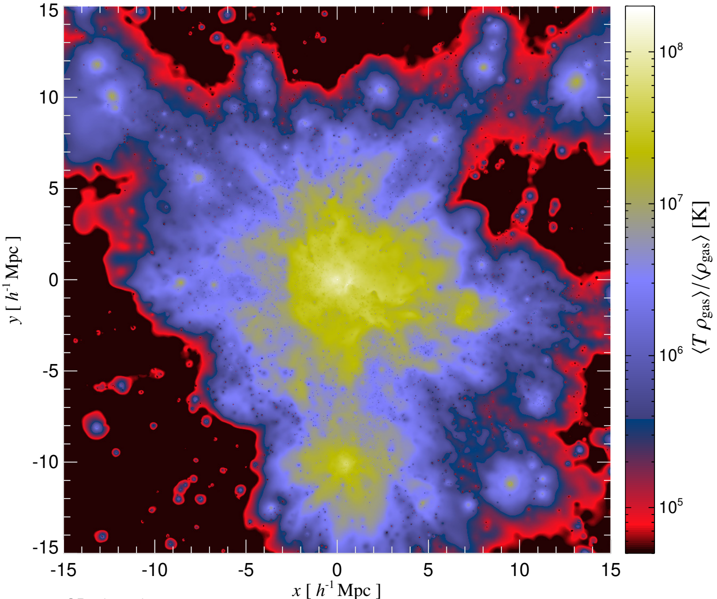
CP, Enßlin, Springel (2008)

Cluster simulation: gas density



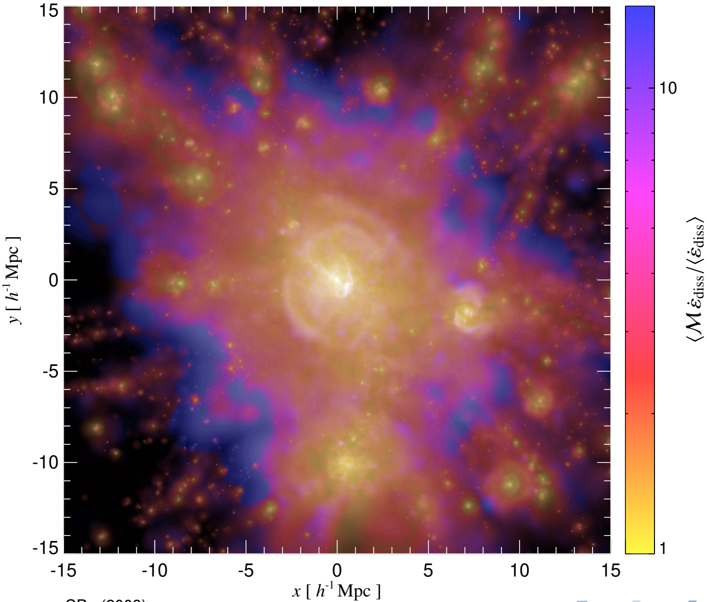
CP+ (2008)

Mass weighted temperature



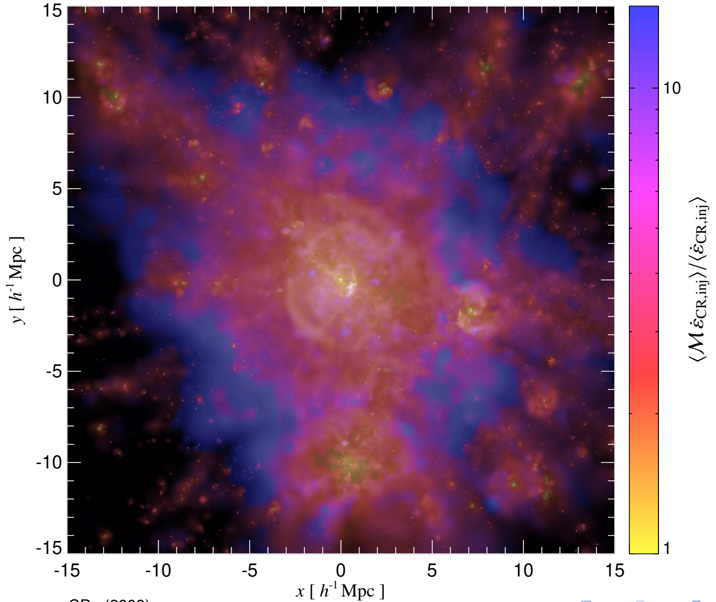
CP+ (2008)

Mach number distribution weighted by $\varepsilon_{\rm diss}$



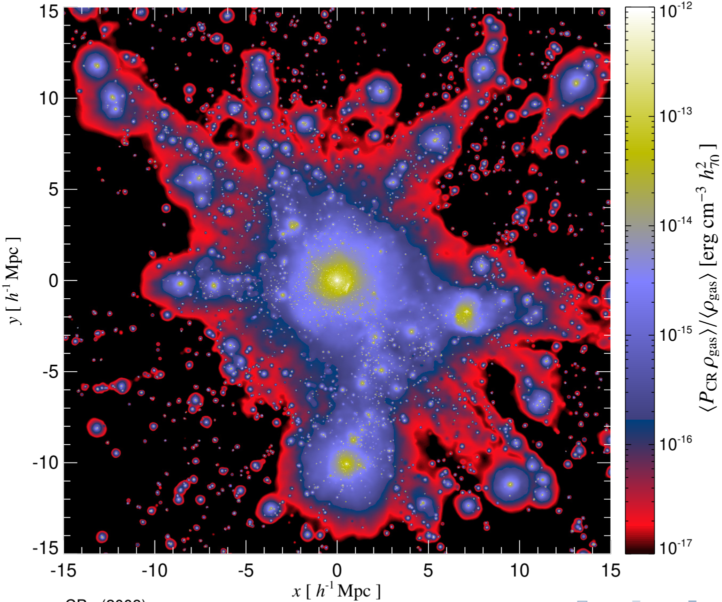
CP+ (2008)

Mach number distribution weighted by $\varepsilon_{\mathrm{CR,inj}}$



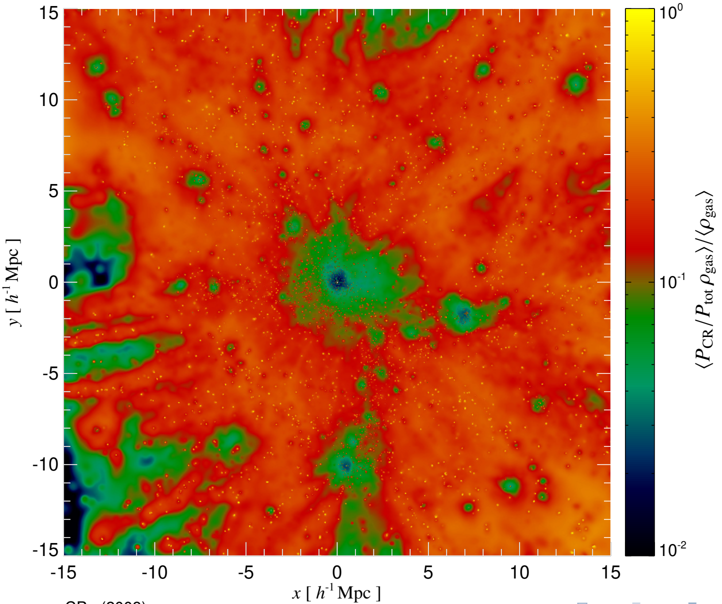
CP+ (2008)

CR pressure P_{CR}



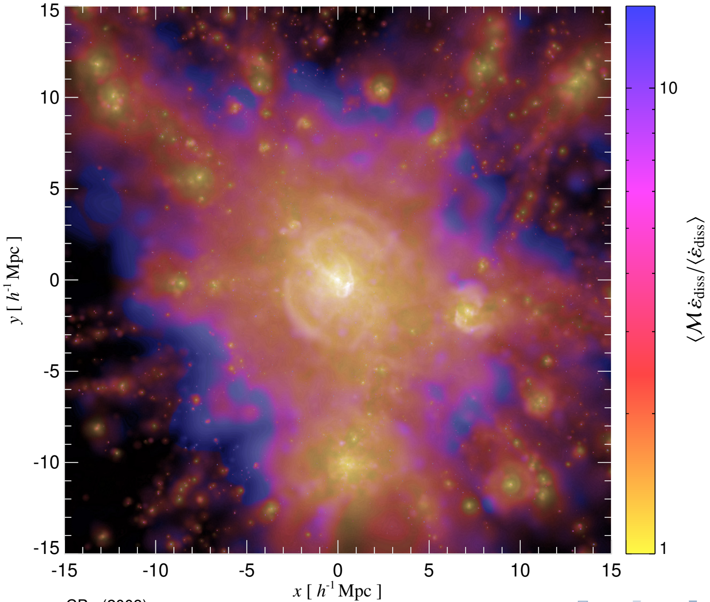
CP+ (2008)

Relative CR pressure P_{CR}/P_{total}



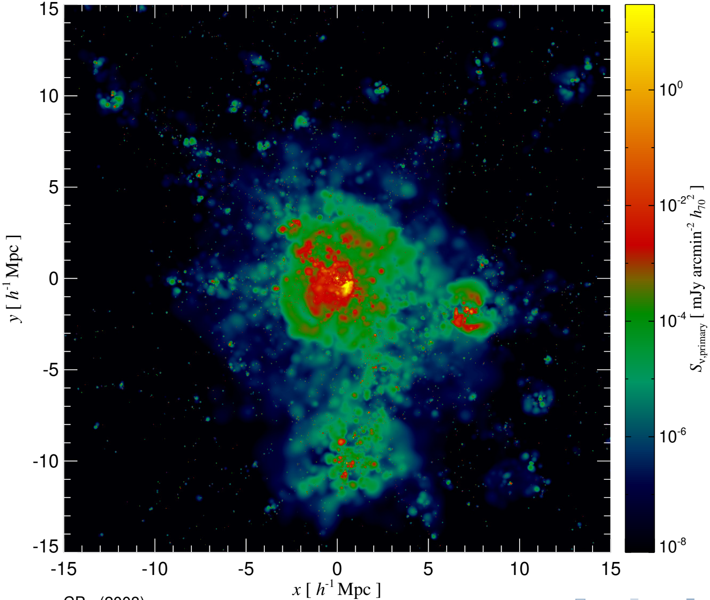
CP+ (2008)

Cosmic web: Mach number



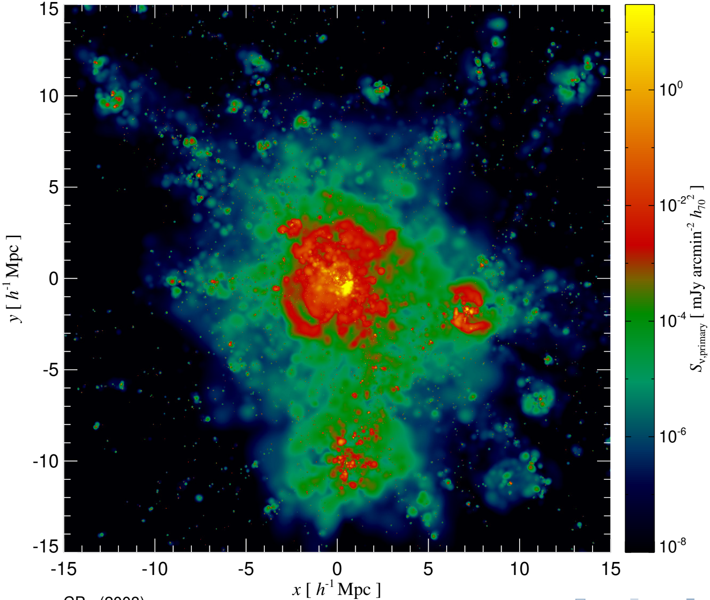
CP+ (2008)

Radio web: primary CRe (1.4 GHz)



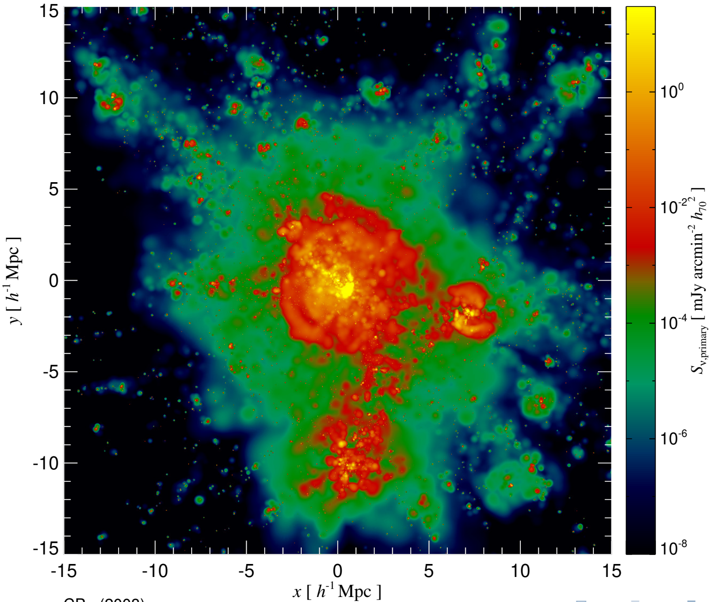
CP+ (2008)

Radio web: primary CRe (150 MHz)



CP+ (2008)

Radio web: primary CRe (15 MHz)



CP+ (2008)

- **Properties of astrophysical magnetic fields:**
  * magnetic fields exist in all astrophysical objects on scales from km to several Mpcs and show field strengths from 10^{-9} G to 10^{15} G
  * magnetized objects include planets, stars, pulsars/magnetars, black-hole accretion discs and jets, galaxies, galaxy clusters
  * magnetic observables: Zeeman effect, synchrotron intensity & polarization, Faraday rotation

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- **Magnetic field evolution:**
  - * Biermann battery can generate $\boldsymbol{B}$ field from a baroclinic flow without $\boldsymbol{B}_0$
  - * the magnetic dynamo stretches, folds, twists, and merges the field so that it grows exponentially fast until saturation
  - * the magnetic flux is frozen into the thermal plasma

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- **Magnetic force and stress:**
  * magnetic pressure causes the fluid to expand perpendicular to the mean magnetic field if $P_B = B^2/8\pi > P_{\text{th}}$
  * magnetic tension aims to reduce the curvature by pulling the field line straight if $P_B > P_{\text{th}}$: analogy of a stretched elastic rubber band!

- **Magneto-hydrodynamic waves and turbulence:**
  * MHD supports 7 modes: two polarization states of Alfvén waves, slow- and fast magnetosonic waves each, and the zero-frequency entropy mode
  * MHD turbulence has an anisotropic cascade where eddies become more elongated towards smaller scales and locally align with $\langle \boldsymbol{B} \rangle$

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- **Magnetic draping:**
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- **Non-thermal processes in clusters:**
  * radio relics and halos prove the existence of volume-filling magnetic fields and relativistic electrons in the intracluster medium
  * the radial extent (short axis) of radio relics that propagate on the sky enables to estimate the magnetic field strength via a cooling length argument
  * what powers radio halos? hadronic interactions or 2$^{\text{nd}}$ order Fermi reacceleration?

Literature for general reading

There are many excellent texts on magnetic fields in the universe. If I had to select three I would probably pick these ones that range from a basic introduction to numerical modeling to a solid review:

- **Introductory text to magneto-hydrodynamics (MHD):**
  Essential Magnetohydrodynamics for Astrophysics, Spruit,
  `https://arxiv.org/abs/1301.5572`
- **Review of numerical techniques for ideal and non-ideal MHD, applied to the context of star formation simulations:**
  Numerical Methods for Simulating Star Formation, Teyssier & Commercon, 2019, FrASS, 6, 51
  `https://arxiv.org/abs/1907.08542`
- **Review of astrophysical magnetic fields with a focus on their generation and maintenance by turbulence:**
  Astrophysical magnetic fields and nonlinear dynamo theory, Brandenburg & Subramanian, 2005, PhR, 417, 1
  `https://arxiv.org/abs/astro-ph/0405052`

If you want to refresh your memory on the derivation of the hydrodynamic equations, of shock waves and hydrodynamic turbulence, I suggest to read Section 3.1 of my

- **Lecture notes that cover many topics in theoretical astrophysics:**
  The Physics of Galaxy Clusters, Pfrommer,
  `https://pages.aip.de/pfrommer/Lectures/clusters.pdf`

Literature for the lecture

Reviews on non-thermal emission from galaxy clusters:

- Brunetti, Jones, *Cosmic Rays in Galaxy Clusters and Their Nonthermal Emission,* 2014, IJMPD, 2330007.
- van Weeren, de Gasperin, Akamatsu, Brüggen, Feretti, Kang, Stroe, Zandanel, *Diffuse Radio Emission from Galaxy Clusters,* 2019, SSRv, 215, 16.

Simulating CRs and non-thermal emission from galaxy clusters:

- Pfrommer, Enßlin, Springel, Jubelgas, Dolag, *Simulating cosmic rays in clusters of galaxies - I. Effects on the Sunyaev-Zel'dovich effect and the X-ray emission,* 2007, MNRAS, 378, 385.
- Pfrommer, Enßlin, Springel, *Simulating cosmic rays in clusters of galaxies - II. A unified scenario for radio halos and relics with predictions of the gamma-ray emission,* 2008, MNRAS, 385, 1211.
- Pfrommer, *Simulating cosmic rays in clusters of galaxies - III. Non-thermal scaling relations and comparison to observations,* 2008, MNRAS, 385, 1242.
- Enßlin, Pfrommer, Miniati, Subramanian, *Cosmic ray transport in galaxy clusters: implications for radio halos, gamma-ray signatures, and cool core heating,* 2011, A&A, 527, 99.

Magnetic draping in galaxy clusters:

- Dursi & Pfrommer, *Draping of Cluster Magnetic Fields over Bullets and Bubbles - Morphology and Dynamic Effects,* 2008, ApJ, 677, 993.
- Pfrommer & Dursi, *Detecting the orientation of magnetic fields in galaxy clusters,* 2010, Nature Phys. 6, 520.